Sharp rate for the dual quantization problem

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January 16, 2012

Abstract

In this paper we establish the sharp rate of the optimal dual quantization problem. The notion of dual quantization was recently introduced in the paper [9], where it has been shown that, at least in an Euclidean setting, dual quantizers are based on a Delaunay triangulation, the dual counterpart of the Voronoi tessellation on which "regular" quantization relies. Moreover, this new approach shares an intrinsic stationarity property, which makes it very valuable for numerical applications.

We establish in this paper the counterpart for dual quantization of the celebrated Zador theorem, which describes the sharp asymptotics for the quantization error when the quantizer size tends to infinity. The proof of this theorem relies among others on an extension of the so-called Pierce Lemma by means of a random quantization argument.

Keywords: quantization, quantization rate, Zador's Theorem, Pierce's Lemma, dual quantization, Delaunay triangulation, random quantization.

MSC: 60F25

1 Introduction

Starting with [8] and continued in [9], we introduced a new notion of vector quantization called *dual quantization* (or *Delaunay quantization* in an Euclidean framework). We developed in [10] some first applications towards the design of numerical schemes for multi-dimensional optimal stopping and stochastic control problems arising in Finance (see also [1]). In general, the principle of dual quantization consists of mapping an $\mathbb{R}^d$-valued random vector (r.v.) onto a nonempty finite subset (or *grid*) $\Gamma \subset \mathbb{R}^d$ using an appropriate random splitting operator $\mathcal{J}_\Gamma : \Omega_0 \times \mathbb{R}^d \to \Gamma$ (defined on an exogenous probability space $(\Omega_0, \mathcal{S}_0, \mathbb{P}_0)$) which satisfies the intrinsic *stationary property*

$$\forall\, \xi \in \operatorname{conv}(\Gamma), \qquad \mathbb{E}_{\mathbb{P}_0}(\mathcal{J}_\Gamma(\xi)) = \int_{\Omega_0} \mathcal{J}_\Gamma(\omega_0, \xi)\, \mathbb{P}_0(d\omega_0) = \xi \tag{1}$$

where $\operatorname{conv}(\Gamma)$ denotes the convex hull of Γ in $\mathbb{R}^d$. Then, for every random vector (r.v.) X defined on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and taking values in $\operatorname{conv}(\Gamma)$ (once canonically extended to $(\Omega_0 \times \Omega, \mathcal{S}_0 \otimes \mathcal{S}, \mathbb{P}_0 \otimes \mathbb{P})$), it holds

$$\mathbb{E}_{\mathbb{P}_0 \otimes \mathbb{P}}(\mathcal{J}_\Gamma(X) \,|\, X) = X.$$

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This means that the resulting approximation $\mathcal{J}_\Gamma(X)$ of X always satisfies a *reverse* stationarity property which can be compared to the more classical stationary property induced by the nearest neighbour projection (which produces the so-called Voronoi quantization). The latter one coincides with $\mathrm{Proj}_\Gamma(X)$ of X onto Γ, namely $\mathbb{E}(X\,|\,\mathrm{Proj}_\Gamma(X)) = \mathrm{Proj}_\Gamma(X)$, but has the drawback that it is only satisfied for exactly optimal grids for the quadratic mean (Voronoi) quantization error (see below) and exclusively in an Euclidean framework.

To both operators, or quantization frameworks, corresponds a functional approximation operator: the Voronoi functional approximation induces the stepwise constant functional approximation operator defined by $f\circ \mathrm{Proj}_\Gamma$ whereas dual quantization leads to an operator defined for every $\xi\in \mathrm{conv}(\Gamma)$ as

$$\mathbb{J}_\Gamma(f)(\xi) = \mathbb{E}_{\mathbb{P}_0}\big(f(J_\Gamma(\omega_0,\xi))\big) = \sum_{x\in\Gamma} f(x)\lambda_x(\xi),$$

where $\lambda_x(\xi) = \mathbb{P}_0(J_\Gamma(.,\xi) = x)$, $x\in\Gamma$, are barycentric "pseudo-coordinates" of ξ in Γ satisfying $\lambda_x(\xi)\in[0,1]$, $\sum_{x\in\Gamma}\lambda_x(\xi) = 1$ and $\sum_{x\in\Gamma}\lambda_x(\xi)x = \xi$. The operator $\mathbb{J}_\Gamma$ is an *interpolation* operator which turns out to be under appropriate conditions more regular (continuous and stepwise affine, see [10]) than the above stepwise constant one. It has been emphasized in [9, 8, 10] how to take advantage of this intrinsic stationary property to produce more accurate cubature formulae for (conditional) expectation approximation regardless of any optimality property of the grid(s) with respect to a r.v. X.
Typically, for every function $f:\mathbb{R}^d\to\mathbb{R}$ having a Lipschitz continuous differential

$$\begin{aligned}\Big|\mathbb{E}_{\mathbb{P}} f(X) - \mathbb{E}_{\mathbb{P}\otimes\ Prob_0} f(J_\Gamma(\omega_0,X))\Big| &\le \Big\|f(X) - \mathbb{E}\big(f(J_\Gamma(\omega_0,X))\,|\,X\big)\Big\|_{L^1(\mathbb{P}\otimes\mathbb{P}_0)}\\ &\le [Df]_{\mathrm{Lip}}\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_0}|X-J_\Gamma(\omega_0,X)|^2\end{aligned}$$

where

$$\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_0}(|X-J_\Gamma(\omega_0,X)|^2\,|\,X) = \sum_{x\in\Gamma}\lambda_x(X)|X-x|^2.$$

More generally if one aims at approximating $\mathbb{E}\big(f(X)\,|\,g(Y)\big)$by its dually quantized counterpart $\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_0\otimes\mathbb{P}_1}\big(f(J_{\Gamma_X}(\omega_0,X))\,|\,J_{\Gamma_Y}(\omega_1,Y)\big)$ (with obvious notations), it is possible to get under natural additional assumptions error bounds based on the two related dual quantization error moduli, see the proof (Step 2) of Proposition 2.1 in [10].
More generally, this leads to investigate the behaviour of

$$\Big\|X - J_\Gamma(\omega_0,X)\Big\|^p_{L^p(\mathbb{P}\otimes\mathbb{P}_0)} = \mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_0}\Big(\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_0}(|X-J_\Gamma(\omega_0,X)|)^p\,|\,X)\Big)$$

so as to make it as small as possible. This program is in fact three-folded:
– the first step is to minimize the above conditional expectation, *i.e.* $\mathbb{E}(|\xi - J_\Gamma(\omega_0,\xi)|)^p$ for every $\xi\in\mathrm{conv}(\Gamma)$, for a fixed grid Γ *i.e.* to determine the *best* splitting random operator J_Γ. In a regular quantization, this phase corresponds to showing that the nearest neighbour projection on Γ is the best projection on Γ.
– The second step is "optional" . Its aim is to find grids which minimizes the mean dual quantization error $\Big\|X - J_\Gamma(\omega_0,X)\Big\|_{L^p(\mathbb{P}\otimes\mathbb{P}_0)}$ among all grids Γ such that $\mathbb{P}(X\in\mathrm{conv}(\Gamma)) = 1$.

In fact these first two phases have already been solved in [9]. The remaining question to be elucidated is the rate of convergence to 0 of the mean dual quantization error – minimized over all grids of size at most N – as N grows to infinity. This third problem is the counterpart of the celebrated Zador's Theorem recalled below. The aim of this paper is to establish a counterpart of this theorem in a dual quantization framework, for L^∞-bounded r.v. but also, once extended in an appropriate way like in [9], to general r.v.'s.

Let us now write things in a more formal way. This new quantization modulus leads to an optimal dual quantization problem

$$d_{n,p}(X) = \inf\Big\{\|F_p(X;\Gamma)\|_p,\ \Gamma\subset\mathbb{R}^d,\ |\Gamma|\le n\Big\},$$

where F_p denotes the local dual quantization error function

$$F_p(\xi;\Gamma) = \inf\left\{\Big(\sum_{x\in\Gamma}\lambda_x\|\xi-x\|^p\Big)^{\frac{1}{p}},\ \lambda_x\in[0,1],\ \sum_{x\in\Gamma}\lambda_x\, x=\xi,\ \sum_{x\in\Gamma}\lambda_x=1\right\}.$$

Since this notion only makes sense for compactly supported r.v. X, we also consider the extension to unbounded r.v. X (see [9]) defined by

$$\bar F_p(\xi;\Gamma) := F_p(\xi;\Gamma)\,\mathbf{1}_{\mathrm{conv}(\Gamma)}(\xi) + \mathrm{dist}(X,\Gamma)\,\mathbf{1}_{\mathrm{conv}(\Gamma)^c}(\xi).$$

and the extended dual quantization error given by

$$\bar d_{n,p}(X) = \inf\Big\{\|\bar F_p(X;\Gamma)\|_p,\ \Gamma\subset\mathbb{R}^d,\ |\Gamma|\le n\Big\}.$$

Recall that the "regular" Voronoi optimal quantization problem reads

$$e_{n,p}(X) = \inf\left\{\left(\mathbb{E}\min_{x\in\Gamma}\|X-x\|^p\right)^{\frac{1}{p}},\ \Gamma\subset\mathbb{R}^d,\ |\Gamma|\le n\right\}.$$

It is well-known that $e_{n,p}(X)\downarrow 0$ as soon as $n\to\infty$ and $X\in L^p(\mathbb{P})$. Moreover, this rate of convergence to 0 of $e_{n,p}(X)$ is ruled by the celebrated Zador Theorem (see [4]).

Theorem 1. *Let $X\in L^{p'}_{\mathbb{R}^d}(\mathbb{P})$, $p'>p$. Assume the distribution $\mathbb{P}_X$ of X is decomposed as $\mathbb{P}_X = h.\lambda_d+\nu$, $\nu\perp\lambda_d$. Then*

$$\lim_{n\to\infty} n^{\frac{1}{d}}e_{n,p}(X) = Q^{vq}_{\|\cdot\|,p,d}\,\|h\|^{\frac{1}{p}}_{\frac{d}{p+d}}$$

where

$$Q^{vq}_{\|\cdot\|,p,d} = \inf_{n\to\infty} n^{\frac{1}{d}}\, e_{n,p}(U([0,1]^d))\in(0,\infty).$$

This rate depending on d is known as the *curse of dimensionality.* Its statement and proof go back to Zador in 1954 for uniform distribution, with an extension to possibly unbounded absolutely continuous distributions by Bucklew and Wise (see [2]). It has been finally established rigourously (as far as mathematical standard are concerned) in [4] in 2000. A comprehensive survey of the history of quantization can be found in [5].

The paper is entirely devoted to the proof of Theorem 2 (Zador' like theorem) and Proposition 7 (Pierce like Lemma). Our global strategy of proof is close to that adopted in [4] for the original Zador Theorem. However, it significantly differs at some points , especially when dealing with the extended modulus $\bar d_{n,p}(X)$. In one dimension the exact rate $O(n^{-1})$ for $d_{n,p}(X)$ and $\bar d_{n,p}(X)$ follows from a random quantization argument detailed in Section 4 which is an extension of the so-called Pierce Lemma $d_{n,p}(X)$ (in fact, we even state a slightly more general result than requested for our purpose). This rate can be transferred in a d-dimensional framework to $O(n^{-\frac{1}{d}})$ using a product (dual) quantization argument (see Section 3.2). Finally the sharp upper bound is obtained in Section 5 by successive approximation procedures of the density of X similar to that developed in [4], whereas the lower bound relies on a new "firewall" Lemma.

Notations: $\mathrm{conv}(A)$ stands for the convex hull of A, $|A|$ for its cardinality and $\lfloor x\rfloor$ will denote the (lower) integral part of the real number x.

2 Main results

The aim of this paper is to prove for any $p > 0$ and any norm on $\mathbb{R}^d$ the counterpart of Zador's Theorem in the framework of dual quantization for both $d_{n,p}$ and $\bar{d}_{n,p}$ error moduli.

Theorem 2. *(a) Let $X \in L^\infty_{\mathbb{R}^d}(\mathbb{P})$. Assume the distribution $\mathbb{P}_X$ of X reads $\mathbb{P}_X = h.\lambda_d + \nu$, $\nu \perp \lambda_d$. Then*

$$\lim_{n\to\infty} n^{\frac{1}{d}}\, d_{n,p}(X) = \lim_{n\to\infty} n^{\frac{1}{d}}\, \bar{d}_{n,p}(X) = Q^{dq}_{\|\cdot\|,p,d}\, \|h\|^{\frac{1}{p}}_{\frac{d}{p+d}}$$

where

$$Q^{dq}_{\|\cdot\|,p,d} = \inf_{n\to\infty} n^{\frac{1}{d}}\, d_{n,p}(U([0,1]^d)) \in (0,\infty).$$

(b) Let $X \in L^{p'}_{\mathbb{R}^d}(\mathbb{P})$, $p' > p$. Assume the distribution $\mathbb{P}_X$ of X reads $\mathbb{P}_X = h.\lambda_d + \nu$, $\nu \perp \lambda_d$. Then

$$\lim_{n\to\infty} n^{\frac{1}{d}}\, \bar{d}_{n,p}(X) = Q^{dq}_{\|\cdot\|,p,d}\, \|h\|^{\frac{1}{p}}_{\frac{d}{p+d}}.$$

(c) If $d = 1$, then

$$d_{n,p}(U([0,1])) = \left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}} \frac{1}{n-1},$$

which implies $Q^{dq}_{|\,.\,|,p,1} = \left(\frac{2^{p+1}}{p+2}\right)^{\frac{1}{p}} Q^{vq}_{|\,.\,|,p,1}$.

Moreover we will also establish in Section 5 a upper bound for the dual quantization coefficient $Q^{\mathrm{dq}}_{\|\cdot\|,p,d}$ when $\|\cdot\| = |\cdot|_{\ell^r}$.

As a step toward the above sharp rate theorem, we will also establish a counterpart of the so-called Pierce Lemma (stated in an operating form in [6] which is very useful for applications since it provides non-asymptotic error bounds which only depend on the moments of the r.v. X (and the size of the optimal grid).

Proposition 1. *Let $r, p \in [1,\infty)$ and $r \le p$. Then it holds for every $d \in \mathbb{N}$*

$$Q^{dq}_{|\cdot|_{\ell^r},p,d} \le d^{\frac{1}{r}} \cdot Q^{dq}_{|\cdot|,p,1}.$$

Since this upper bound achieves the same asymptotic rate as in the case of regular quantization (cf. Cor. 9.4 in [4]), we believe the rate of $d^{\frac{1}{r}}$ to be also the true one for $Q^{\mathrm{dq}}_{\|\cdot\|,p,d}$ as $d \to \infty$.

Proposition 2 (d-dimensional extended Pierce Lemma)**.** *Let p, $\eta > 0$. There exists an integer $n_{d,p,\eta} \ge 1$ and a real constant $C_{d,p,\eta}$ such that, for every $n \ge n_{p,\eta}$ and every random variable $X \in L^{p+\eta}_{\mathbb{R}^d}(\Omega_0, \mathcal{A}, \mathbb{P})$,*

$$\bar{d}_{n,p}(X) \le C_{d,p,\eta}\sigma_{p+\eta,\|\cdot\|}(X)\, n^{-1/d}.$$

where $\sigma_{p+\eta,\|\cdot\|}(X) = \inf_{a\in\mathbb{R}^d} \|X - a\|_{L^{p+\eta}}$.
If $\mathrm{supp}(\mathbb{P}_X)$ is compact then the same inequality holds true for $d_{n,p}(X)$.

3 Dual quantization: definition and basic properties

3.1 Definitions

Assume $\mathbb{R}^d$ equipped with a norm $\|\cdot\|$. First we recall the definition of the regular quantization problem for a random vector (r.v.) $X : (\Omega, \mathcal{S}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}^d)$ and

Definition 1. *Let $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$ and $\Gamma \subset \mathbb{R}^d$.*

1. *We define the L^p-mean regular quantization error for a grid Γ as*

$$e_p(X;\Gamma) = (\mathbb{E} \min_{x\in\Gamma} \|X-x\|^p)^{1/p} = \|d(X,\Gamma)\|_{L^p}.$$

2. *The optimal regular quantization error, which can be achieved by a grid Γ of size not exceeding n is given by*

$$e_{n,p}(X) = \inf\big\{e_p(X;\Gamma) : \Gamma \subset \mathbb{R}^d, |\Gamma| \le n\big\}.$$

Following [9], the dual quantization error can be introduced as follows.

Definition 2. *Let $X \in L^p(\mathbb{P})$ and $\Gamma \subset \mathbb{R}^d$.*

1. *We define the local p-dual quantization error for a grid Γ as*

$$F_p(\xi;\Gamma) = \inf\Big\{\Big(\sum_{x\in\Gamma} \lambda_x \|\xi - x\|^p\Big)^{\frac{1}{p}} : \lambda_x \in [0,1],\ \sum_{x\in\Gamma}\lambda_x x = \xi,\ \sum_{x\in\Gamma}\lambda_x = 1\Big\}.$$

2. *The L^p-mean dual quantization error for X induced by a grid Γ is then given by*

$$\begin{aligned} d_p(X;\Gamma) &= \|F_p(X;\Gamma)\|_{L^p} \\ &= \Bigg(\mathbb{E}\inf\Big\{\sum_{x\in\Gamma}\lambda_x\|\xi - x\|^p : \lambda_x \in [0,1] \ \textit{and}\ \sum_{x\in\Gamma}\lambda_x x = \xi, \sum_{x\in\Gamma}\lambda_x = 1\Big\}\Bigg)^{1/p}. \end{aligned}$$

3. *The optimal dual quantization error, which can be achieved by a grid Γ of size not exceeding n will be denoted by*

$$d_{n,p}(X) = \inf\big\{d_p(X;\Gamma) : \Gamma \subset \mathbb{R}^d, |\Gamma| \le n\big\}.$$

4. *The extended L^p-mean dual quantization error induces by a grid Γ is defined by*

$$\bar{d}_p(X,\Gamma) = \Big\|F_p(X;\Gamma)\,\mathbf{1}_{\mathrm{conv}(\Gamma)}(X) + \mathrm{dist}(X,\Gamma)\,\mathbf{1}_{\mathrm{conv}(\Gamma)^c}(X)\Big\|_{L^p}.$$

5. *The optimal extended dual quantization error, which can be achieved by a grid Γ of size not exceeding n will be denoted by*

$$\bar{d}_{n,p}(X) = \inf\big\{\bar{d}_p(X;\Gamma) : \Gamma \subset \mathbb{R}^d, |\Gamma| \le n\big\}.$$

Remarks. 1. Since the above quantities only depend on the distribution of the r.v. X we will also write $d_p(\mathbf{P},\Gamma)$ for $d_p(X,\Gamma)$ and $d_{n,p}(\mathbf{P})$ for $d_{n,p}(X)$ where $\mathbf{P} = \mathbb{P}_X$.

2. To alleviate notations, we will use throughout the paper F^p, d^p and $\bar{d}^p$, …instead of F_p^p, d_p^p and $\bar{d}_p^p$,…

In fact the terminology *dual quantization* refers to a canonical example of intrinsic stationary splitting operator: the dual quantization operator.

To be more precise, assume $\mathbb{R}^d$ is equipped with a norm $\|\,.\,\|$ and let $p \in [1,+\infty)$. Let $\Gamma = \{x_1,\dots,x_n\} \subset \mathbb{R}^d$ be a grid of size $n \ge d+1$ such that $\mathrm{aff.dim}(\Gamma) = d$.
The idea is to "split" $\xi \in \mathrm{conv}(\Gamma)$ among at most $d+1$ affinely independent points in Γ (which convex hull contains ξ) proportionally to its barycentric coordinates. There are usually many possible choices so we introduced a minimal inertia based criterion to select the most appropriate

"neighbours" of ξ, namely the function $F_p(\xi;\Gamma)$ defined for every ξ as the value of the minimization problem

$$F_p(\xi;\Gamma) = \inf_{(\lambda_1,\dots,\lambda_n)} \left\{ \left(\sum_{i=1}^{n} \lambda_i \|\xi - x_i\|^p\right)^{\frac{1}{p}}, \lambda_i \in [0,1], \sum_i \lambda_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} \xi \\ 1 \end{bmatrix} \right\}.$$

Owing to the compactness of constraint set, there exists at least one solution $\lambda^*(\xi)$ and for any such solution, one shows using convex extremality arguments that the set $I^*(\xi) := \{i \in I \text{ s.t. } \lambda_i^*(\xi) > 0\}$ defines an affinely independent subset $\{x_i,\ i \in I^*(\xi)\}$.

When this solution is always unique, the dual quantization operator is simply defined on $\mathrm{conv}(\Gamma)$ by

$$\forall\, \xi \in \mathrm{conv}(\Gamma),\ \forall\, \omega_0 \in \Omega_0, \quad \mathcal{J}_\Gamma^*(\omega_0,\xi) = \sum_{i \in I(\xi)^*} x_i \mathbf{1}_{\{\sum_{j=1}^{i-1} \lambda_j^*(\xi) \le U(\omega_0) < \sum_{j=1}^{i} \lambda_j^*(\xi)\}}.$$

Thus in the quadratic ($p = 2$) Euclidean case and when Γ is in the so-called "general position" ([1]), then $\Big\{\{\xi \text{ s.t. } I^*(\xi) = I\},\ |I| \le d+1\Big\}$ makes up a Borel partition of $\mathrm{conv}(\Gamma)$ (with possibly empty elements), known in 2-dimension as the *Delaunay triangulation* of Γ (see [12] for the connection to Delaunay triangulations). In a general framework, we refer to [9] for a construction of dual quantization operators.

These operators play the role of the nearest neighbour projections for "regular" Voronoi quantization and one checks that

$$\begin{aligned} \|\mathcal{J}_\Gamma^*(X) - X\|_{L^p(\mathbb{P}_0\otimes\mathbb{P})} &= \|F_p(X;\Gamma)\|_{L^p(\mathbb{P})} \\ &= \mathbb{E} \inf \left\{ \left(\sum_{i=1}^{n} \lambda_i \|X - x_i\|^p\right)^{\frac{1}{p}}, \lambda_i \ge 0, \sum_i \lambda_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

The second step of the optimization process is to find (at least) one grid which optimally "fits" (the distribution of) X *i.e.* which is solution to the second level optimization problem

$$d_{n,p}(X) = \inf \left\{ \|\mathcal{J}_\Gamma^*(X) - X\|_{L^p(\mathbb{P}_0\otimes\mathbb{P})},\ \mathcal{J}_\Gamma^* : \Omega_0 \times \mathrm{conv}(\Gamma) \to \Gamma, \mathrm{conv}(\Gamma) \supset \mathrm{supp}\mathbb{P}_X,\ |\Gamma| \le n \right\}.$$

Note that if $X \in L^\infty(\mathbb{P})$, $d_{n,p}(X) < +\infty$ iff $n \ge d+1$ (and is identically infinite if X is not essentially bounded). The existence of an optimal grid (or dual quantizer) has been established in [9] as well as the following characterization $d_{n,p}(X)$ as the lowest L^p-mean approximation error by r.v. taking at most n values and satisfying the intrinsic stationary property *i.e.*

$$d_{n,p}(X) = \inf \left\{ \|X - \widehat{X}\|_{L^p(\mathbb{P}_0\otimes\mathbb{P})},\ |\widehat{X}(\Omega_0 \times \Omega)| \le n,\ \mathbb{E}_{\mathbb{P}_0\otimes\mathbb{P}}(\widehat{X} \mid X) = X \right\}.$$

A stochastic optimization procedure based on a stochastic gradient approach has been devised a in [9] to compute optimal grids w.r.t. various distributions (so far, uniform over $[0,1]^2$, normal, $(W_1, \sup_{t\in[0,1]} W_t)$, W standard Brownian motion).

When a random vector X is not essentially bounded, the above approach cannot be developed since no finite grid can contain its support. In that case, we need to extend the definition of our splitting operator $\mathcal{J}_\Gamma$ outside the convex hull of Γ. One way to proceed (see [9]) is to consider again a (deterministic) nearest neighbour projection Proj_Γ

$$\forall\, \xi \in \mathbb{R}^d \setminus \mathrm{conv}(\Gamma), \quad \mathcal{J}_\Gamma(\omega_0,\xi) = \mathrm{Proj}_\Gamma(\xi).$$

[1] no $d+2$ points of Γ lie on a sphere.

An alternative could have been $\mathcal{J}_\Gamma(\omega_0,\xi) = \mathrm{Proj}_{\mathrm{conv}(\Gamma)}(\xi)$. We loose the intrinsic stationary property, however we were able to show the existence of an optimal grid solution to the resulting minimization problem

$$\bar{d}_{n,p}(X) = \inf\left\{\|F_p(X;\Gamma)\mathbf{1}_{\{X\in\mathrm{conv}(\Gamma)\}} + \mathrm{dist}(X,\Gamma)\mathbf{1}_{\{X\notin\mathrm{conv}(\Gamma)\}}\|_{L^p(\mathbb{P})}\right\}.$$

It is clear that $d_{n,p}(X)$ and $\bar{d}_{n,p}(X)$ do not coincide even for bounded r.v. but one can show that

$$\min(d_{n,p}(X),\bar{d}_{n,p}(X)) \geq e_{n,p}(X)$$

where $e_{n,p}(X)$ is the "regular" Voronoi L^p-mean quantization error at level n defined by

$$e_{n,p}(X) = \inf\left\{\|X - \mathrm{Proj}_\Gamma(X)\|_{L^p(\mathbb{P})},\ |\Gamma|\leq n\right\}.$$

The above dual quantization problem is characterized in terms of best L^p-approximation by the following theorem established in [9].

Theorem 3. *Let $X \in L^0(\Omega,\mathcal{S},\mathbb{P})$ and $n\in\mathbb{N}$. Then*

$$\begin{aligned} d_{n,p}(X) &= \inf\big\{\mathbb{E}\|X - \mathcal{J}_\Gamma(X)\|_{L^p} : \mathcal{J}_\Gamma : \Omega_0\times\mathbb{R}^d\to\Gamma,\ \textit{intrinsic stationary},\\ &\qquad\qquad \mathrm{supp}(\mathbb{P}_X)\subset\mathrm{conv}(\Gamma),\ |\Gamma|\leq n\big\}\\ &= \inf\big\{\mathbb{E}\|X-\widehat{X}\|_{L^p} : \widehat{X} : (\Omega_0\times\Omega,\mathcal{S}_0\otimes\mathcal{S},\mathbb{P}_0\otimes\mathbb{P})\to\mathbb{R}^d,\\ &\qquad\qquad |\widehat{X}(\Omega_0\times\Omega)|\leq n,\ \mathbb{E}(\widehat{X}|X)=X\big\}\leq+\infty. \end{aligned}$$

These quantities are finite iff $X\in L^\infty(\Omega,\mathcal{S},\mathbb{P})$.

As already mentioned, we established in [9] the existence of dual quantizers at level $n\in\mathbb{N}$ for the L^p-norm when $p\in(1,\infty)$. We recall this result her (without proof) for the reader's convenience.

Theorem 4 (Existence of optimal quantizers)**.** *Let $X\in L^p(\mathbb{P})$ for some $p\in(1,\infty)$.*

(a) If $\mathrm{supp}(\mathbb{P}_X)$ is compact, then there exists for every $n\in\mathbb{N}$ a grid $\Gamma_n^*\subset\mathbb{R}^d$, $|\Gamma_n^*|\leq n$ such that $d_p(X;\Gamma_n^*) = d_{n,p}(X)$.

(b) If $\mathbb{P}_X$ is strongly continuous in the sense that it assigns mass zero to all hyperplanes in $\mathbb{R}^d$, then there exists for every $n\in\mathbb{N}$ a grid $\Gamma_n^*\subset\mathbb{R}^d$, $|\Gamma_n^*|\leq n$ such that $\bar{d}_p(X;\Gamma_n^*) = \bar{d}_{n,p}(X)$.

If furthermore $|\mathrm{supp}(\mathbb{P}_X)|\geq n$, then the above statements hold with $|\Gamma_n^*| = n$.

3.2 Local properties of the dual quantization functional

We establish in this paragraph some general properties for the local dual quantization functional F^p, which will be needed for the final proof of Theorem 2.

Proposition 3. *Let Γ_1, $\Gamma_2\subset\mathbb{R}^d$ be finite grids and let $\xi\in\mathbb{R}^d$. Then*

$$\Gamma_1\subset\Gamma_2 \Longrightarrow F_p(\xi;\Gamma_2)\leq F_p(\xi;\Gamma_1).$$

Proof. Assume $\Gamma_1 = \{x_1,\dots,x_m\}$ and $\Gamma_2 = \{x_1,\dots,x_m,x_{m+1},\dots,x_n\}$. Then

$$\begin{aligned} F^p(\xi;\Gamma_2) = \min_{\substack{\lambda\in\mathbb{R}^n\\ \text{s.t. } \left[\begin{smallmatrix} x_1 & \dots & x_n\\ 1 & \dots & 1\end{smallmatrix}\right]\lambda = \left[\begin{smallmatrix}\xi\\1\end{smallmatrix}\right],\, \lambda\geq 0}} \sum_{i=1}^n \lambda_i\,\|\xi - x_i\|^p &\leq \min_{\substack{\lambda=(\lambda_1,0)\\ \text{s.t. } \left[\begin{smallmatrix} x_1 & \dots & x_m\\ 1 & \dots & 1\end{smallmatrix}\right]\lambda_1 = \left[\begin{smallmatrix}\xi\\1\end{smallmatrix}\right],\, \lambda_1\geq 0}} \sum_{i=1}^n \lambda_i\,\|\xi - x_i\|^p\\ &= \min_{\substack{\lambda\in\mathbb{R}^m\\ \text{s.t. } \left[\begin{smallmatrix} x_1 & \dots & x_m\\ 1 & \dots & 1\end{smallmatrix}\right]\lambda = \left[\begin{smallmatrix}\xi\\1\end{smallmatrix}\right],\, \lambda\geq 0}} \sum_{i=1}^m \lambda_i\,\|\xi - x_i\|^p = F^p(\xi;\Gamma_1). \end{aligned}$$

□

Moreover, we will make use of the following three properties established in [9].

Proposition 4 (Scalar bound). *Let $\Gamma = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ with $x_1 \le \ldots \le x_n$. Then*

$$\forall \xi \in [x_1, x_n], \quad F^p(\xi;\Gamma) \le \max_{1\le i\le n-1} \Big(\frac{x_{i+1}-x_i}{2}\Big)^p.$$

Proposition 5 (Local product Quantization). *Let $\|\cdot\| = |\cdot|_{\ell^p}$ defined for every $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$ by $|\xi|_{\ell^p} = \Big(\sum_{1\le i\le d} |\xi^i|^p\Big)^{1/p}$ and $\Gamma = \prod_{j=1}^d \Gamma_j$ for some $\Gamma_j \subset \mathbb{R}$. Then*

$$F_p(\xi;\Gamma) = \left(\sum_{j=1}^d F^p(\xi^j, \Gamma_j)\right)^{\frac{1}{p}}.$$

One then may derive in the next proposition a first upper bound for the asymptotics of the optimal dual quantization error of distributions with bounded support when the size of the grid tends to infinity.

Proposition 6 (Product Quantization). *Let $C = a + \ell[0,1]^d$, $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, $\ell > 0$, be a hypercube, parallel to the coordinate axis with common edge length l. Let Γ be the product quantizer of size $(m+1)^d$ defined by*

$$\Gamma = \prod_{k=1}^d \Big\{a_j + \frac{i\ell}{m},\ i = 0, \ldots, m\Big\}.$$

There exists a positive real constant $C_{d,\|\cdot\|}$ such that

$$\forall \xi \in C, \quad F^p(\xi;\Gamma) \le C_{d,\|\cdot\|} \cdot \Big(\frac{l}{2}\Big)^p \cdot m^{-p}. \tag{2}$$

4 Extended Pierce lemma and applications

The aim of this section is to provide a non-asymptotic upper-bound for the optimal dual quantization error in the spirit of [11], which achieves nevertheless the optimal rate of convergence when the size n goes to infinity. Like for Voronoi quantization this upper-bound deeply relies on a random quantization argument. In fact, it can be established for a (slightly) more general family of error functionals than the ones considered so far for dual and regular quantization.

4.1 One dimensional extended Pierce Lemma

Let

$$\mathcal{I}_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n,\ -\infty < x_1 \le x_2 \le \cdots \le x_n < +\infty\}$$

be the set of "non-decreasing" n-tuples of $\mathbb{R}^n$.

Definition 3. *Let $(\Omega_0, \mathcal{A})$ be a measurable space and let $n \ge 1$ be an integer. A measurable functional $\Phi_n : (\Omega_0 \times \mathcal{I}_n \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}or\mathcal{I}_n \otimes \mathcal{B}or(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}or(\mathbb{R}))$ is called a* splitting functional *at level n if it satisfies:*

$\forall\, (x_1, \ldots, x_n) \in \mathcal{I}_n,\ \forall\, \xi \in \mathbb{R}$,

$$\begin{cases} (i) & \xi \in [x_i, x_{i+1}],\ i = 1, \ldots, n-1 & \Longrightarrow & \forall\, \omega \in \Omega_0,\ \Phi_n(\omega, x_1, \ldots, x_n, \xi) \in [x_i, x_{i+1}], \\ (ii) & \xi \in (-\infty, x_1] & \Longrightarrow & \forall\, \omega \in \Omega_0,\ \Phi_n(\omega, x_1, \ldots, x_n, \xi) = x_1, \\ (iii) & \xi \in [x_n, \infty) & \Longrightarrow & \forall\, \omega \in \Omega_0,\ \Phi_n(\omega, x_1, \ldots, x_n, \xi) = x_n \end{cases}$$

EXAMPLES: (a) *Nearest neighbour/Voronoi quantization.* For every $i = 1, \ldots, n-1$, let $x_{i+\frac{1}{2}} := \frac{x_i + x_{i+1}}{2}$. Set

$$\Phi_n(\omega, x_1, \ldots, x_n, \xi) = \begin{cases} x_i & \text{if } \xi \in [x_i, x_{i+\frac{1}{2}}), \\ \in \{x_i, x_{i+1}\} & \text{if } \xi = x_{i+\frac{1}{2}}, \\ x_{i+1} & \text{if } \xi \in (x_{i+\frac{1}{2}}, x_{i+1}]. \end{cases}$$

(b) *Dual quantization.* $\Omega_0 = [0, 1]$ and

$$\begin{aligned} \Phi_n(\omega, x_1, \ldots, x_n, \xi) \quad = \quad & \sum_{i=1}^{n-1} \Big(x_i \mathbf{1}_{\{\omega \le \frac{x_{i+1} - \xi}{x_{i+1} - x_i}\}} \\ & + x_{i+1} \mathbf{1}_{\{\frac{\xi - x_i}{x_{i+1} - x_i} \ge x_n\}} \Big) \mathbf{1}_{[x_i, x_{i+1})}(\xi) + x_1 \mathbf{1}_{\{\xi < x_1\}}(\xi) + x_n \mathbf{1}_{\{\xi \ge x_n\}}(\xi). \end{aligned}$$

It follows from (i) that a splitting functional at level n satisfies for every $p > 0$, $\omega \in \Omega_0$, $(x_1, \ldots, x_n) \in \mathcal{I}_n$, $\xi \in \mathbb{R}$,

$$d(\xi, \{x_1, \ldots, x_n\})^p \le |\xi - \Phi_n(\omega, x_1, \ldots, x_n, \xi)|^p \le A_{p,n}(x_1, \ldots, x_n, \xi)^p \tag{3}$$

where

$$A_{p,n}(x_1, \ldots, x_n, \xi) = \left(\sum_{i=1}^{n-1} (x_{i+1} - x_i)^p \mathbf{1}_{\{x_i \le \xi < x_{i+1}\}} + (\xi - x_n)^p \mathbf{1}_{\{\xi \ge x_n\}} + (x_1 - \xi)^p \mathbf{1}_{\{\xi < x_1\}} \right)^{\frac{1}{p}}.$$

Let X be random variable defined on $(\Omega_0, \mathcal{A}, \mathbb{P})$.

$$X \in L^p(\mathbb{P}) \Longrightarrow X - \Phi_n(., x_1, \ldots, x_n, X),\ A(x_1, \ldots, x_n, X) \in L^p(\mathbb{P}).$$

Furthermore, it follows from (3) that

$$\inf_{(x_1, \ldots, x_n) \in \mathcal{I}_n} \|A_{p,n}(x_1, \ldots, x_n, X)\|_{L^p} \ge \inf_{(x_1, \ldots, x_n) \in \mathcal{I}_n} \|X - \Phi_n(., x_1, \ldots, x_n, X)\|_{L^p} \ge e_{n,p}(X).$$

The functionals $A_{p,n}$ share two important properties extensively used in what follows:

- *Consistency* : if, for every $(x_1, \ldots, x_n) \in \mathcal{I}_n$ and for every $\xi \in \mathbb{R}$,

  $$\forall\, i \in \{1, \ldots, n-1\},\ A_{p,n}(x_1, \ldots, x_n, \xi) = A_{p,n+1}(x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n, \xi).$$

  As a straightforward consequence, it follows that

  $$n \mapsto \inf_{(x_1, \ldots, x_n) \in \mathcal{I}_n} \|A_{p,n}(x_1, \ldots, x_n, X)\|_{L^p} \text{ is non-increasing.} \tag{4}$$

- *Scaling*: $\forall\, \omega \in \Omega_0$, $\forall\, (x_1, \ldots, x_n) \in \mathcal{I}_n$, $\forall\, \xi \in \mathbb{R}$, $\forall\, \alpha \in \mathbb{R}_+$, $\forall\, \beta \in \mathbb{R}$

  $$\begin{aligned} A_{p,n}(\alpha\, x_1 + \mu, \ldots, \alpha\, x_n + \beta, \xi) \quad &= \quad \alpha\, A_{p,n}(x_1, \ldots, x_n, \xi), \\ A_{p,n}(x_1, \ldots, x_n, -\xi) \quad &= \quad A_{p,n}(-x_n, \ldots, -x_1, \xi). \end{aligned}$$

The main result of this section shows the existence of a universal non-asymptotic upper bound for the error induced by splitting functionals which appears as an extension of the so-called Pierce Lemma established in [4] (see also [6]) as crucial step towards Zador's Theorem for regular Voronoi quantization.

Theorem 5. *Let p, $\eta > 0$. There exists a positive real constant $C_{p,\eta} > 0$ and an integer $n_{p,\eta} \geq 1$ such that for any random variables $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ and any sequence of splitting functionals $(\Phi_n)_{n\geq 1}$ defined on a probability space $(\Omega_0, \mathcal{A}_0, \mathbb{P}_0)$*

$$\forall\, n \geq n_{p,\eta}, \quad \inf_{(x_1,\ldots,x_n)\in\mathcal{I}_n} \|X - \Phi_n(.,x_1,\ldots,x_n,X)\|_{L^p} \leq C_{p,\eta}\|X\|_{L^{p+\eta}(\mathbb{P}_0\otimes\mathbb{P})}n^{-1}$$

(where X and Φ_n have been canonically extended to $\Omega_0 \times \Omega$).

Proof. Step 1. We first assume that X is $[1, +\infty)$-valued. Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. Pareto(δ)-distributed random variables (with probability density $f_{_Y}(y) = \delta y^{\delta-1}\mathbf{1}_{\{y\geq 1\}}$) defined on a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$. By considering $\widetilde{\Omega} = \Omega \times \Omega'$, $\widetilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A}'$, $\widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$, one may assume without loss of generality that X and the sequence $(Y_n)_{n\geq 1}$ are independent (and defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$). For convenience we will denote by $\|\,.\,\|_{L^p}$ the L^p-norm on $(\Omega_0 \times \Omega, \mathcal{A}_0 \otimes \mathcal{A}, \mathbb{P}_0 \otimes \mathbb{P})$ throughout the proof.

Let $\delta = \delta(p, \eta) \in (0, \frac{p}{\eta})$, let $\ell = \ell(p, \eta) = \frac{p}{\delta(p,\eta)}$. For every $n \geq \ell(p, \eta)$, let $\widetilde{n} = n - \ell + 2$.

$$\begin{aligned}
\inf_{(x_1,\ldots,x_n)\in\mathcal{I}_n} \|X - \Phi_n(.,x_1,\ldots,x_n,X)\|_{L^p} &\leq \inf_{(x_1,\ldots,x_n)\in\mathcal{I}_n} \|A_{p,n}(x_1,\ldots,x_n,X)\|_{L^p}\\
&\leq \inf_{(1,x_2,\ldots,x_{\widetilde{n}})\in\mathcal{I}_{\widetilde{n}}} \|A_{p,\widetilde{n}}(1,x_2,\ldots,x_{\widetilde{n}},X)\|_{L^p}\\
&\leq \|A_{p,\widetilde{n}}(1,Y_1^{(n)},\ldots,Y_{\widetilde{n}}^{(n)},X)\|_{L^p}
\end{aligned}$$

where, for every $n \geq 1$, $Y^{(n)} = (Y_1^{(n)}, \ldots, Y_n^{(n)})$ denotes the standard order statistics of the first n terms of the sequence $(Y_n)_{n\geq 1}$. For notational convenience we set $Y_0^{(n)} = 1$. Then, using that X and $(Y_k)_{k\geq 1}$ are independent, we get

$$\begin{aligned}
\mathbb{E}\, A_{p,\widetilde{n}}(1,Y_1^{(n)},\ldots,Y_{\widetilde{n}}^{(n)},X)^p \;\leq\; & \sum_{i=0}^{n-\ell} \mathbb{E}\Big(\big(Y_{i+1}^{(n)} - Y_i^{(n)}\big)^p \mathbf{1}_{\{X\in[Y_i^{(n)},Y_{i+1}^{(n)})\}}\Big)\\
&+\mathbb{E}\Big(\big(X - Y_{n-\ell+1}^{(n)}\big)^p \mathbf{1}_{\{X\geq Y_{n-\ell+1}^{(n)}\}}\Big).
\end{aligned}$$

Step 3. Now we will compute the successive terms of the above sum. Set $\kappa = p + \eta$. Let $i \in \{1, \ldots, n - \ell\}$.

$$\begin{aligned}
\mathbb{E}\Big(\big(Y_{i+1}^{(n)} - Y_i^{(n)}\big)^p \mathbf{1}_{\{X\in[Y_i^{(n)},Y_{i+1}^{(n)})\}}\Big) &\leq \mathbb{E}\Big(\big(Y_{i+1}^{(n)} - Y_i^{(n)}\big)^p \mathbf{1}_{\{X\geq Y_i^{(n)}\}}\Big)\\
&\leq \mathbb{E}\Big(\big(Y_{i+1}^{(n)} - Y_i^{(n)}\big)^p (Y_i^{(n)})^{-\kappa}\Big)\mathbb{E}\, X^\kappa
\end{aligned}$$

where we used that X and $(Y_k)_{k\geq 1}$ are independent. Now, denoting by $F_{_Y}(u) = (1 - u^{-\delta})\mathbf{1}_{\{y\geq 1\}}$ the distribution function of the Pareto(δ)-distribution, elementary computations show that

$$\begin{aligned}
&\mathbb{E}\Big(\big(Y_{i+1}^{(n)} - Y_i^{(n)}\big)^p (Y_i^{(n)})^{-\kappa}\Big)\\
&\quad= \int_{1\leq u\leq v} du\, dv\, u^{-\kappa}(v-u)^p F_{_Y}(u)^{i-1}(1 - F_{_Y}(u))^{n-i-1} f_{_Y}(u) f_{_Y}(v) \frac{n!}{(i-1)!(n-i-1)!}\\
&\quad= \delta B(n - i + \frac{\eta}{\delta}, i) B((n-i)\delta - r, r+1) \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i)}\\
&\quad= \delta\, \frac{\Gamma(n - i + \frac{\eta}{\delta})\Gamma(i)}{\Gamma(n + \frac{\eta}{\delta})} \frac{\Gamma((n-i)\delta - r)\Gamma(p+1)}{\Gamma((n-i)\delta + 1)} \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i)}
\end{aligned}$$

where the functions

$$\Gamma(t) = \int_0^\infty u^{t-1}e^{-u}d,\ t>0 \ \text{ and } \ B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du,\ a,\, b>0$$

are known to satisfy $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

One checks likewise that the above equality still holds for $i=0$.

Then one derives

$$\begin{aligned}\sum_{i=0}^{n-\ell}\mathbb{E}\Big(\big(Y_{i+1}^{(n)}-Y_i^{(n)}\big)^p(Y_i^{(n)})^{-\kappa}\Big) &= \delta\Gamma(p+1)\frac{\Gamma(n+1)}{\Gamma(n+\frac{\eta}{\delta})}\sum_{i=0}^{n-\ell}\frac{\Gamma(n-i+\frac{\eta}{\delta})\Gamma((n-i)\delta-r)}{\Gamma((n-i)\delta+1)\Gamma(n-i)}\\ &= \delta\Gamma(p+1)\frac{\Gamma(n+1)}{\Gamma(n+\frac{\eta}{\delta})}\sum_{i=\ell}^{n}\frac{\Gamma(i+\frac{\eta}{\delta})\Gamma(i\delta-r)}{\Gamma(i\delta+1)\Gamma(i)}.\end{aligned}$$

Using that, for every $a>0$, $\dfrac{\Gamma(x+a)}{\Gamma(x)} \sim x^a$ as $x\to\infty$, we derive that

$$\frac{\Gamma(i+\frac{\eta}{\delta})}{\Gamma(i)} \sim i^{\frac{\eta}{\delta}},\qquad \frac{\Gamma(i\delta-r)}{\Gamma(i\delta+1)} \sim (i\delta-r)^{-(p+1)} \sim (i\delta)^{-(p+1)} \quad \text{as} \quad i\to\infty$$

and

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{\eta}{\delta})} \sim n^{1-\frac{\eta}{\delta}} \quad \text{as} \quad n\to+\infty.$$

Consequently,

$$\begin{aligned}\sum_{i=0}^{n-\ell}\mathbb{E}\Big(\big(Y_{i+1}^{(n)}-Y_i^{(n)}\big)^p(Y_i^{(n)})-\kappa\Big) &\sim n^{1-\frac{\eta}{\delta}}\sum_{i=1}^{n}\delta^{-(p+1)}\frac{1}{i^{p+1-\frac{\eta}{\delta}}}\\ &= \delta^{-(p+1)}n^{-p}\frac{1}{n}\sum_{i=\ell}^{n}\left(\frac{i}{n}\right)^{\frac{\eta}{\delta}-r-1}\\ &\sim \delta^{-(p+1)}n^{-p}\underbrace{\int_0^1 u^{\frac{\eta}{\delta}-r-1}du}_{<+\infty \text{ since } \frac{\eta}{\delta}>p} \ \text{ as } \ n\to\infty\end{aligned}$$

so that

$$\sum_{i=0}^{n-\ell}\mathbb{E}\Big(\big(Y_{i+1}^{(n)}-Y_i^{(n)}\big)^p(Y_i^{(n)})^{-\kappa}\Big) \sim C_{p,\eta}n^{-p} \quad \text{as} \quad i\to\infty.$$

The remaining term can be treated as follows.

$$\begin{aligned}\mathbb{E}\Big(\big(X-Y_{n-\ell+1}^{(n)}\big)^p\mathbf{1}_{\{X\geq Y_{n-\ell+1}^{(n)}\}}\Big) &\leq \mathbb{E}\,X^p\mathbf{1}_{\{X\geq Y_{n-\ell+1}^{(n)}\}}\\ &\leq \mathbb{E}\,X^p\frac{X^\eta}{(Y_{n-\ell+1}^{(n)})^\eta}\\ &= \mathbb{E}\,X^\kappa\,\mathbb{E}(Y_{n-\ell+1}^{(n)})^{-\eta}.\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}(Y_{n-\ell+1}^{(n)})^{-\eta} &= \frac{\Gamma(n+1)}{\Gamma(n-\ell+1)\Gamma(\ell)}\int_0^1(1-v)^{n-\ell}v^{\ell+\frac{\eta}{\delta}-1}dv\\ &= \frac{\Gamma(n+1)}{\Gamma(n-\ell+1)\Gamma(\ell)}\frac{\Gamma(n-\ell+1)\Gamma(\ell+\frac{\eta}{\delta})}{\Gamma(n+\frac{\eta}{\delta})}\\ &\sim \frac{\Gamma(\ell+\frac{\eta}{\delta})}{\Gamma(\ell)}n^{-\frac{\eta}{\delta}} = o(n^{-p})\end{aligned}$$

since $\frac{\eta}{\delta} > r$. Finally

$$\limsup_n n^p \left(\sum_{i=0}^{n-\ell} \mathbb{E}\Big((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \Big) + \mathbb{E}(Y_{n-\ell+1}^{(n)})^{-\eta} \right) < +\infty$$

so that

$$\sup_{n \geq \ell(p,\eta)} n^p \left(\sum_{i=0}^{n-\ell} \mathbb{E}\Big((Y_{i+1}^{(n)} - Y_i^{(n)})^p (Y_i^{(n)})^{-\kappa} \Big) + \mathbb{E}(Y_{n-\ell+1}^{(n)})^{-\eta} \right) < +\infty.$$

This shows that for every $n \geq n_{p,\eta} := \ell(p,\eta)$,

$$\inf_{(x_1,\ldots,x_n)\in\mathcal{I}_n} \|A_{p,n}(x_1,\ldots,x_n,X)\|_{L^p} \leq C_{p,\eta} \frac{\|X\|_{L^{p+\eta}}^{1+\frac{\eta}{p}}}{n}.$$

STEP 4. If X is a non-negative random variable, applying the second step to $X+1$ and using the scaling property satisfied by $A_{p,n}$ yields for $n \geq n_{p,\eta}$ (as defined in Step 3),

$$\begin{aligned} \inf_{(1,x_2,\ldots,x_n)\in\mathcal{I}_n} \|A_{p,n}(1,x_2,\ldots,x_n,X)\|_{L^p} &= \inf_{(0,x_2,\ldots,x_n)\in\mathcal{I}_n} \|A_{p,n}(x_1,\ldots,x_n,X+1)\|_{L^p} \\ &\leq C_{p,\eta} \frac{\|1+X\|_{L^{p+\eta}}^{1+\frac{\eta}{p}}}{n} \\ &\leq C'_{p,\eta} \frac{(1+\|X\|_{L^{p+\eta}}^{1+\frac{\eta}{p}})}{n}. \end{aligned}$$

We may assume that $\|X\|_{L^{p+\eta}} \in (0,\infty)$. Then, applying the above bound to the non-negative random variable $\widetilde{X} = \frac{X}{\|X\|_{L^{p+\eta}}}$ yields using positive homogeneity

$$\inf_{(0,x_2,\ldots,x_n)\in\mathcal{I}_n} \|A_{p,n}(0,x_2,\ldots,x_n,X)\|_{L^p} \leq \|X\|_{L^{p+\eta}} C'_{p,\eta} \frac{1+1}{n}.$$

STEP 5. Let X be a real-valued random variable and let for every integer $n \geq 1$, $x_1,\ldots,x_n \in (-\infty,0)$, $x_{n+1} = 0$ and $x_{n+2},\ldots,x_{2n+1} \in (0,+\infty)$. It follows that

$$\begin{aligned} A_{p,2n+1}(x_1,\ldots,x_{2n+1},X)^p &= A_{p,n+1}(x_1,\ldots,x_{n+1},X_+)^p \mathbf{1}_{\{X\geq 0\}} \\ &\quad + A_{p,n+1}(x_1,\ldots,x_{n+1},-X_-)^p \mathbf{1}_{\{X<0\}} \\ &= A_{p,n+1}(x_1,\ldots,x_{n+1},X_+)^p + A_{p,n+1}(-x_{n+1},\ldots,-x_1,X_-)^p. \end{aligned}$$

Consequently, if $p \geq 1$, we get using that $u^{\frac{1}{p}} + v^{\frac{1}{p}} \leq (u+v)^{\frac{1}{p}}$, u, $v \geq 0$,

$$\begin{aligned} \inf_{(x_1,\ldots,x_{2n+1})\in\mathcal{I}_{2n+1}} \|A_{p,2n+1}(x_1,\ldots,x_{2n+1},X)\|_{L^p} \leq & \inf_{(0,x_2,\ldots,x_{n+1})\in\mathcal{I}_{n+1}} \|A_{p,n+1}(0,x_2,\ldots,x_{n+1},X_+)\|_{L^p} \\ & + \inf_{(0,x_2,\ldots,x_{n+1})\in\mathcal{I}_{n+1}} \|A_{p,n}(0,x_2,\ldots,x_{n+1},X_-)\|_{L^p}. \end{aligned}$$

Hence, it follows from Step 3 that, for every $n \geq n_{p,\eta} - 1$,

$$\begin{aligned} \inf_{(x_1,\ldots,x_{2n+1})\in\mathcal{I}_{2n+1}} \|A_{p,2n+1}(x_1,\ldots,x_{2n+1},X)\|_{L^p} &\leq C'_{p,\eta} \Big(\|X_-\|_{L^{p+\eta}} + \|X_+\|_{L^{p+\eta}} \Big) \frac{1}{n+1}, \\ &= C'_{p,\eta} \|X\|_{L^{p+\eta}} \frac{1}{n+1} \end{aligned}$$

where we used that $\|X\|_{L^{p+\eta}} \leq \|X_-\|_{L^{p+\eta}} + \|X_+\|_{L^{p+\eta}}$. Finally, the monotonicity property (4) implies that, for every $n \geq 2\,n_{p,\eta}$,

$$\inf_{(x_1,\ldots,x_n)\in\mathcal{I}_n} \|A_{p,n}(x_1,\ldots,x_n,X)\|_{L^p} \leq 2C'_{p,\eta} \frac{\|X\|_{L^{p+\eta}}}{n}.$$

If $p \in (0,1)$, one obtains using directly (5) that

$$\inf_{(x_1,\ldots,x_{2n+1})\in\mathcal{I}_{2n+1}} \|A_{p,2n+1}(x_1,\ldots,x_{2n+1},X)\|_{L^p}^p \le (C'_{p,\eta})^p\Big(\|X_-\|_{p+\eta}^p + \|X_+\|_{p+\eta}^p\Big)\frac{1}{(n+1)^p}.$$

Now

$$\|X_-\|_{p+\eta}^p + \|X_+\|_{L^{p+\eta}}^p \le (\|X_-\|_{L^{p+\eta}}^{p+\eta} + \|X_+\|_{L^p p+\eta}^{p+\eta})^{\frac{p}{p+\eta}} = \|X\|_{L^{p+\eta}}^p$$

so that the conclusion remains the same. □

4.2 A d-dimensional non-asymptotic upper-bound for the dual quantization error

Now, using Proposition 5, we are in position to show Proposition 7 (the d-dimensional version of the extended Pierce Lemma) which provides a non-asymptotic upper-bound at the exact rate for dual quantization error moduli.

Proposition 7 (d-dimensional extended Pierce Lemma)**.** *Let p, $\eta > 0$. There exists an integer $n_{d,p,\eta} \ge 1$ and a real constant $C_{d,p,\eta}$ such that, for every $n \ge n_{p,\eta}$ and every random variable $X \in L^{p+\eta}_{\mathbb{R}^d}(\Omega_0,\mathcal{A},\mathbb{P})$,*

$$\bar d_{n,p}(X) \le C_{d,p,\eta}\sigma_{p+\eta,\|\cdot\|}(X)\, n^{-1/d}.$$

where $\sigma_{p+\eta,\|\cdot\|}(X) = \inf_{a\in\mathbb{R}^d}\|X-a\|_{L^{p+\eta}}$.
If $\mathrm{supp}(\mathbb{P}_X)$ is compact then the same inequality holds true for $d_{n,p}(X)$.

Proof. First note that $\bar d_{n,p}(X) = \bar d_{n,p}(X-a)$, $a \in \mathbb{R}^d$ (invariance by translation) so we may assume that X is $L^{p+\eta}$-centered *i.e.* $\sigma_{p+\eta,\|\cdot\|}(X) = \|X\|_{L^{p+\eta}}$.
$\rhd$ $d = 1$. In this one dimensional setting one may consider only ordered n-tuples $\gamma = (x_1,\ldots,x_n)$. One derives from Theorem 5 and the example (b) that follows that, for every $n \ge n_{p,\eta}$,

$$\bar d_{n,p}(X) = \inf_{\gamma\in\mathcal{I}_n}\|X - \Phi_n^{dq}(U,\gamma)\|_{-p} \le C_{p,\eta}\|X\|_{L^{p+\eta}}n^{-1}$$

where $U \sim U([0,1])$ is independent of X.

$\rhd$ $d \ge 2$. Let γ be an optimal quantizer of size $n_1\cdots n_d \le n$. Then if $X = (X^1,\ldots,X^d)$ denotes the components of X, one has if $\min_\ell n_\ell \ge n_{p,\eta}$ (from the one dimensional case) using Proposition 5

$$\begin{aligned}
\bar d^p_{n,p}(X,\Gamma_\gamma) &= \mathbb{E}\,\bar F_{n,p}(X;\gamma)\\
&\le C_{d,\|\cdot\|,p}\sum_{\ell=1}^d \mathbb{E}\,\bar F_{n,p}(X^\ell;n_\ell)\\
&\le C_{d,\|\cdot\|,p}C_{p,\eta}\sum_{\ell=1}^d \|X^\ell\|_{L^{p+\eta}}n_\ell^{-1}\\
&\le C_{d,\|\cdot\|,p,\eta}\max_{1\le\ell\le d}\|X^\ell\|_{L^{p+\eta}} \times d\,\lfloor n^{\frac1d}\rfloor^{-1}\\
&\le C'_{d,\|\cdot\|,p,\eta}\big\|\,\|X\|\,\big\|_{L^{p+\eta}}n^{-\frac1d}.
\end{aligned}$$

□

5 Proof of the sharp rate theorem

On the way to proof the sharp rate theorem, we have to establish few additional propositions.

Proposition 8 (Sub-linearity). *Let* $\mathbf{P} = \sum_{i=1}^m s_i \mathbf{P}_i$, $\sum_{i=1}^m s_i = 1$ *and* $\sum_{i=1}^m n_i \le n$. *Then*

$$d_{n,p}^p(\mathbf{P}) \le \sum_{i=1}^m s_i\, d_{n_i,p}^p(\mathbf{P}_i).$$

Proof. For $\varepsilon > 0$ and every $i = 1, \dots, m$, let $\Gamma_i \subset \mathbb{R}^d$, $|\Gamma_i| \le n_i$ such that

$$d^p(\mathbf{P}_i; \Gamma_i) \le (1+\varepsilon)\, d_{n_i}^p(\mathbf{P}_i).$$

Then by Proposition 3 and with $\Gamma = \bigcup_{i=1}^m \Gamma_i$

$$\begin{aligned} d_{n,p}^p(\mathbf{P}) &\le d_{n,p}^p(\mathbf{P}; \Gamma) \\ &= \sum_{i=1}^m s_i \int F_p^p(\xi; \Gamma)\, d\mathbf{P}_i(\xi) \\ &\le \sum_{i=1}^m s_i \int F_p^p(\xi; \Gamma_i)\, d\mathbf{P}_i(\xi) \\ &\le (1+\varepsilon) \sum_{i=1}^m s_i\, d_{n_i,p}^p(\mathbf{P}_i), \end{aligned}$$

so that sending $\varepsilon \to 0$ yields the assertion. □

Remark. Proposition 8 does not hold for $\bar{d}_n^p$, which causes substantial difficulties in the proof of the sharp rate compared to the regular quantization setting.

Proposition 9 (Scaling property). *Let* $C = a + \rho[0,1]^d$ *be a* d*-dimensional hypercube, parallel to the coordinate axis, with edge length* $\rho > 0$. *Then*

$$d_{n,p}(\mathcal{U}(C)) = \rho \cdot d_{n,p}(\mathcal{U}([0,1]^d)).$$

Proof. We have

$$\begin{aligned} d^p(\mathcal{U}(C); \{a+\rho x_1, \dots, a+\rho x_n\}) &= \int_{[0,\rho]^d} \min_{\substack{\lambda \in \mathbb{R}^n \\ \text{s.t. } \left[\begin{smallmatrix} \rho x_1 & \dots & \rho x_n \\ 1 & \dots & 1 \end{smallmatrix}\right]\lambda = \left[\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right],\, \lambda \ge 0}} \sum_{i=1}^n \lambda_i \,\|\xi - \rho x_i\|^p \;\frac{d\lambda^d(\xi)}{\lambda^d([0,\rho]^d)} \\ &= \int_{[0,1]^d} \min_{\substack{\lambda \in \mathbb{R}^n \\ \text{s.t. } \left[\begin{smallmatrix} \rho x_1 & \dots & \rho x_n \\ 1 & \dots & 1 \end{smallmatrix}\right]\lambda = \left[\begin{smallmatrix} \rho\xi \\ 1 \end{smallmatrix}\right],\, \lambda \ge 0}} \sum_{i=1}^n \lambda_i \,\|\rho\xi - \rho x_i\|^p \; d\lambda^d(\xi) \\ &= \rho^p \int_{[0,1]^d} \min_{\substack{\lambda \in \mathbb{R}^n \\ \text{s.t. } \left[\begin{smallmatrix} x_1 & \dots & x_n \\ 1 & \dots & 1 \end{smallmatrix}\right]\lambda = \left[\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right],\, \lambda \ge 0}} \sum_{i=1}^n \lambda_i \,\|\xi - x_i\|^p \; d\lambda^d(\xi) \\ &= \rho^p \cdot d^p(\mathcal{U}([0,1]^d); \{x_1, \dots, x_n\}), \end{aligned}$$

which yields the assertion. □

The following Lemma shows that also for $\bar{d}_{n,p}$ the convex hull spanned by a sequence of "semi-optimal" quantizers asymptotically covers the interior of $\operatorname{supp}(\mathbb{P}_X)$, a fact which is trivial for $d_{n,p}$ and compact support.

Lemma 1. *Let $K = \mathrm{conv}\{a_1, \dots, a_k\} \subset \overbrace{\mathrm{supp}(\mathbf{P})}^{\circ}$ be a set with $\mathring{K} \neq \emptyset$ and let Γ_n be a sequence of quantizers such that $\bar{d}_{n,p}(\mathbf{P}, \Gamma_n) \to 0$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$K \subset \mathrm{conv}(\Gamma_n).$$

Proof. Set $a_0 = \frac{1}{k} \sum_{i=1}^{k} a_i$ and define for $\rho > 0$

$$\tilde{K}(\rho) = \mathrm{conv}\{\tilde{a}_1(\rho), \dots, \tilde{a}_k(\rho)\} \quad \text{with} \quad \tilde{a}_i = a_0 + (1+\rho)(a_i - a_0).$$

Since $K \subset \overbrace{\mathrm{supp}(\mathbf{P})}^{\circ}$ there exists a $\rho_0 > 0$ such that $\tilde{K} = \tilde{K}(\rho_0) \subset \mathrm{supp}(\mathbb{P}_X)$. We then also denote $\tilde{a}_i(\rho_0)$ by $\tilde{a}_i$. Since moreover $\tilde{a}_i \in \mathrm{supp}(\mathbb{P}_X)$, there exists a sequence $(a_i^n)_{n\geq 1}$ having values in $\mathrm{conv}(\Gamma_n)$ and converging to $\tilde{a}_i$. Otherwise there would be $\varepsilon_0 > 0$ and a subsequence (n') such that $B(\tilde{a}_i, \varepsilon_0) \subset \mathrm{conv}(\Gamma_{n'})^c$. Then $\bar{d}_{n',p}(X, \Gamma_{n'}) \geq \left(\frac{\varepsilon_0}{2}\right)^p \mathbf{P}(B(\tilde{a}_i, \varepsilon_0/2))$ since $\tilde{a}_i \in \mathrm{supp}(P)$ which contradicts the assumption on the sequence $(\Gamma_n)_{n\geq 1}$.
Since K has a nonempty interior, it follows that $\mathrm{aff.dim}\{a_1, \dots, a_k\} = \mathrm{aff.dim}\{\tilde{a}_1, \dots, \tilde{a}_k\} = d$. Consequently, we may choose a set $I^* \subset \{1, \dots, k\}$, $|I^*| = d+1$ so that $\{\tilde{a}_j : j \in I^*\}$ is an affinely independent system in $\mathbb{R}^d$ and furthermore there exists a $n_0 \in \mathbb{N}$ such that the same holds for $\{a_j^n : j \in I^*\}$ and every $n \geq n_0$. Hence, we may write for $n \geq n_0$

$$\tilde{a}_i = \sum_{j\in I^*} \mu_j^{n,i} a_j^n, \quad \sum_{j\in I^*} \mu_j^{n,i} = 1, \quad i = 1, \dots, k. \tag{5}$$

This linear system has the unique asymptotic solution $\mu_j^{\infty,i} = \delta_{ij}$ (Kronecker symbol), which implies $\mu_j^{n,i} \to \delta_{ij}$ for $n \to \infty$.
Now let $\xi \in K \subset \tilde{K}$ and write

$$\xi = \sum_{i=1}^{k} \lambda_i a_i \text{ for some } \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1.$$

One easily verifies that it also holds

$$\xi = \sum_{i=1}^{k} \tilde{\lambda}_i \tilde{a}_i \ \text{ for } \ \tilde{\lambda}_i = \frac{\rho_0}{k(1+\rho_0)} + \frac{\lambda_i}{1+\rho_0} \geq \frac{\rho_0}{k(1+\rho_0)} > 0 \quad \text{and} \quad \sum_{i=1}^{k} \tilde{\lambda}_i = 1,$$

and we furthermore may choose a $n_1 \geq n_0$ such that for every $n \geq n_1$

$$\mu_i^{n,i} > \frac{1}{2} \quad \text{and} \quad |\mu_j^{n,i}| \leq \frac{\rho_0}{4k(1+\rho_0)} \ \forall j \neq i.$$

Using (5) this leads to

$$\xi = \sum_{j\in I^*} \Big(\sum_{i=1}^{k} \tilde{\lambda}_i \mu_j^{n,i} \Big) a_j^n$$

and

$$\sum_{i=1}^{k} \tilde{\lambda}_i \mu_j^{n,i} > \tilde{\lambda}_j \mu_j^{n,j} - \sum_{i=1, i\neq j}^{k} \tilde{\lambda}_i |\mu_j^{n,i}| > \frac{\rho_0}{2k(1+\rho_0)} - \frac{\rho_0}{4k(1+\rho_0)} = \frac{\rho_0}{2k(1+\rho_0)} > 0, \quad j \in I^*.$$

Thus, noting that

$$\sum_{j\in I^*} \sum_{i=1}^{k} \tilde{\lambda}_i \mu_j^{n,i} = \sum_{i=1}^{k} \tilde{\lambda}_i \sum_{j\in I^*} \mu_j^{n,i} = 1$$

finally completes the proof. □

As already said, Proposition 8 does not hold anymore for $\bar{d}_{n,p}$. As a consequence we have to establish an asymptotic firewall Lemma, which will help us in the sequel to overcome this problem also in the non-compact setting.

Lemma 2 (Firewall)**.** *Let $K \subset \mathbb{R}^d$ be compact and convex with $\mathring{K} \neq \emptyset$. Moreover, let $\varepsilon > 0$ such that*

$$K_\varepsilon = \{x \in K : \mathrm{dist}(x, K^c) \geq \varepsilon\} \neq \emptyset.$$

Denote by $\Gamma_{\alpha,\varepsilon}$ a subset of the lattice $\alpha\mathbb{Z}^d$ with edge length $\alpha > 0$ satisfying

$$K \setminus K_\varepsilon \subset \mathrm{conv}(\Gamma_{\alpha,\varepsilon})$$

and for every $x \in K \setminus K_\varepsilon$, $\mathrm{dist}(x, \Gamma_{\alpha,\varepsilon}) \leq C_{\|\cdot\|}\alpha$ where $C_{\|\cdot\|} > 0$ is real constant which only depends on the norm $\|\cdot\|$.
Then, for every grid $\Gamma \subset \mathbb{R}^d, \eta \in (0,1)$ and $\xi \in K_\varepsilon$, it holds

$$F^p(\xi;\Gamma) \geq \frac{1}{(1+\eta)^p} F^p(\xi; (\Gamma \cap \mathring{K}) \cup \Gamma_{\alpha,\varepsilon}) - \Big(1 + \frac{1}{\eta}\Big)^p \tilde{C}_{\|\cdot\|}\alpha^p.$$

Remark. An almost minimal choice for $\Gamma_{\alpha,\varepsilon}$ is

$$\Gamma_{\alpha,\varepsilon} = \alpha\mathbb{Z}^d \cap (K \setminus K_\varepsilon) + \{0, \pm\alpha e^i \, i = 1, \ldots, d\}$$

where $(e^1, \ldots, e^d)$ denotes the canonical basis of $\mathbb{R}^d$.

Proof. Let $\Gamma = \{x_1, \ldots, x_n\}$ and let $\xi \in K_\varepsilon$. Then we may choose $I = I(\xi) \subset \{1, \ldots, n\}$, $|I| \leq d+1$ such that

$$F^p(\xi;\Gamma) = \sum_{i\in I} \lambda_j \|\xi - x_i\|^p, \quad \sum_{i \in I} \lambda_i x_i = \xi,\ \lambda_i \geq 0, \sum_{i\in I} \lambda_i = 1.$$

Assume now that there is a $i_0 \in I$ such that $x_{i_0} \in \Gamma \setminus \mathring{K}$ and $\lambda_{i_0} > 0$ (otherwise the assertion is trivial). Note that there are at most d such components in $I(\xi)$ and choose $\theta = \theta(i_0) \in (0,1)$ such that

$$\tilde{x}_{i_0} = \xi + \theta(x_{i_0} - \xi) \in K \setminus K_\varepsilon.$$

Setting

$$\tilde{\lambda}^0_i = \frac{\lambda_i \theta}{\theta + \lambda_{i_0}(1-\theta)},\ i \in I \setminus \{i_0\}, \quad \tilde{\lambda}^0_{i_0} = \frac{\lambda_{i_0}}{\theta + \lambda_{i_0}(1-\theta)}$$

we arrive at

$$\tilde{\lambda}^0_{i_0}\tilde{x}_{i_0} + \sum_{i \in I\setminus\{i_0\}} \tilde{\lambda}^0_i x_i = \xi,\ \tilde{\lambda}^0_i \geq 0, \sum_{i\in I} \tilde{\lambda}^0_i = 1$$

so that

$$\begin{aligned}
\tilde{\lambda}^0_{i_0}\|\xi - \tilde{x}_{i_0}\|^p + \sum_{j \in I\setminus\{i_0\}} \tilde{\lambda}^0_i \|\xi - x_i\|^p &= \frac{\lambda^0_{i_0}\theta^p}{\theta + \lambda^0_{i_0}(1-\theta)} \|\xi - x_{i_0}\|^p \\
&\quad + \sum_{i \in I\setminus\{i_0\}} \frac{\lambda^0_i \theta}{\theta + \lambda^0_{i_0}(1-\theta)} \|\xi - x_i\|^p \\
&\leq \frac{\theta}{\theta + \lambda_{i_0}(1-\theta)} \sum_{i\in I} \lambda^0_i \|\xi - x_i\|^p \\
&< F^p(\xi;\Gamma)
\end{aligned} \tag{6}$$

where we used that $\theta^p \le \theta$ since $p \ge 1$. Repeating the procedure (at most d times) for every $x_i \in \Gamma \setminus \mathring{K}$ finally yields by induction the existence of $\tilde{x}_i \in K \setminus K_\varepsilon$ and $\tilde{\lambda}_i$, $i \in I$ such that

$$\sum_{i \in I: x_i \notin \mathring{K}} \tilde{\lambda}_i \tilde{x}_i + \sum_{i \in I: x_i \in \mathring{K}} \tilde{\lambda}_i x_i = \xi,\ \tilde{\lambda}_i \ge 0,\ \sum_{i \in I} \lambda_i = 1$$

and

$$F^p(\xi; \Gamma) > \sum_{i \in I: x_i \notin \mathring{K}} \tilde{\lambda}_i \|\xi - \tilde{x}_i\|^p + \sum_{i \in I: x_i \in \mathring{K}} \tilde{\lambda}_i \|\xi - x_i\|^p.$$

Let us denote $\Gamma_{\alpha,\varepsilon} = \{a_1, \ldots, a_m\}$ and let $\tilde{x}_{i_0}$ be a "modified" $x_{i_0} (\in \Gamma \setminus \mathring{K})$. By construction $\tilde{x}_{i_0} \in K \setminus K_\varepsilon \subset \mathrm{conv}(\Gamma_{\alpha,\varepsilon})$ and there is $J_{i_0} \subset \{1, \ldots, m\}$ such that

$$F^p(\tilde{x}_{i_0}, \Gamma_{\alpha,\varepsilon}) = \sum_{j \in J_{i_0}} \mu_j^{i_0} \|\tilde{x}_{i_0} - a_j\|^p,\ \sum_{j \in J_{i_0}} \mu_j^{i_0} x_j = \tilde{x}_{i_0},\ \mu_j^{i_0} \ge 0,\ \sum_{j \in J_{i_0}} \mu_j^{i_0} = 1$$

and

$$\forall\, j \in J_{i_0}, \quad \|\tilde{x}_{i_0} - a_j\| \le C_{\|\cdot\|}\, \alpha$$

for a real constant $C_{\|\cdot\|} > 0$, which only depends on the norm $\|\cdot\|$.
A possible explicit construction (when $\Gamma_{\alpha,\varepsilon}$ is as in the above remark) is the following: one may select $\underline{k} \in \mathbb{Z}^d$ such that $\alpha \underline{k}$ is the nearest neighbour of $\tilde{x}_{i_0}$ in $\Gamma_{\alpha,\varepsilon} \cap K \setminus K_\varepsilon$. Then there exists $\varepsilon_1^{j_0}, \ldots, \varepsilon_d^{j_0} \in \{\pm 1\}$ such that

$$\tilde{x}_{i_0} \in \mathrm{conv}(\alpha \underline{k}, \alpha \underline{k} + \varepsilon_j^{i_0} e^j).$$

The resulting index set J_{i_0} clearly satisfies the above claim.
Using the elementary inequality

$$\forall \eta > 0,\ \forall\, u, v \ge 0, \quad (u + v)^p \le (1 + \eta)^p u^p + \Big(1 + \frac{1}{\eta}\Big)^p v^p,$$

we conclude for every $j \in J_{i_0}$

$$\begin{aligned}\|\xi - a_j\|^p &\le \big(\|\xi - \tilde{x}_{i_0}\| + \|\tilde{x}_{i_0} - a_j\|\big)^p \\ &\le (1 + \eta)^p \|\xi - \tilde{x}_{i_0}\|^p + \Big(1 + \frac{1}{\eta}\Big)^p C_{\|\cdot\|}^p\, \alpha^p.\end{aligned}$$

As a consequence,

$$\sum_{j \in J_{i_0}} \mu_j^{i_0} \|\xi - a_j\|^p \le (1 + \eta)^p \|\xi - \tilde{x}_{i_0}\|^p + \Big(1 + \frac{1}{\eta}\Big)^p C_{\|\cdot\|}^p\, \alpha^p$$

which in turn implies

$$\|\xi - \tilde{x}_{i_0}\|^p \ge \frac{1}{(1 + \eta)^p} \sum_{j \in J_{i_0}} \mu_j^{i_0} \|\xi - a_j\|^p - \Big(1 + \frac{1}{\eta}\Big)^p C_{\|\cdot\|}^p\, \alpha^p.$$

Plugging this inequality in (6) yields

$$\begin{aligned}F^p(\xi; \Gamma) > &\sum_{i \in I: x_i \in \mathring{K}} \tilde{\lambda}_j \|\xi - x_i\|^p \\ &+ \frac{1}{(1 + \eta)^p} \sum_{i \in I: x_i \notin \mathring{K}} \tilde{\lambda}_i \sum_{j \in J_i} \mu_j^i \|\xi - a_j\|^p \\ &- d\Big(1 + \frac{1}{\eta}\Big)^p C_{\|\cdot\|}^p\, \alpha^p \\ \ge &\frac{1}{(1 + \eta)^p} F^p\big(\xi; (\Gamma \setminus \{x_{j_0}\}) \cup \Gamma_{\alpha,\varepsilon}\big) - \Big(1 + \frac{1}{\eta}\Big)^p \tilde{C}_{\|\cdot\|}^p\, \alpha^p\end{aligned}$$

where $\tilde{C}^p_{\|\cdot\|} = dC^p_{\|\cdot\|} > 0$. □

Now we can establish the sharp rate for the uniform distribution $U([0,1]^d)$.

Proposition 10 (Uniform distribution)**.** *For every $p \geq 1$,*

$$Q^{dq}_{\|\cdot\|,p,d} := \inf_{n\geq 0} n^{1/d}\, d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) = \lim_{n\to\infty} n^{1/d}\, d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big).$$

Proof. Let $n, m \in \mathbb{N}$, $m < n$ and set $k = k(n,m) = \left\lfloor \left(\frac{n}{m}\right)^{1/d} \right\rfloor$.
Covering the unit hypercube $[0,1]^d$ by k^d translates $C_1, \ldots, C_{k^d}$ of the hypercube $\left[0, \frac{1}{k}\right]^d$, we arrive at $\mathcal{U}\big([0,1]^d\big) = k^{-d} \sum_{i=1}^{k^d} \mathcal{U}(C_i)$. Hence, Proposition 8 yields

$$d^p_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) \leq k^{-d} \sum_{i=1}^{k^d} d^p_m(\mathcal{U}(C_i)).$$

Furthermore, Proposition 9 states

$$d_{m,p}(\mathcal{U}(C_i)) = k^{-1}\, d_{m,p}\big(\mathcal{U}\big([0,1]^d\big)\big),$$

so that we may conclude for all $n, m \in \mathbb{N}$, $m < n$,

$$d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) \leq k^{-1}\, d_{m,p}\big(\mathcal{U}\big([0,1]^d\big)\big).$$

Thus, we get

$$\begin{aligned} n^{1/d}\, d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) &\leq k^{-1}\, n^{1/d}\, d_{m,p}\big(\mathcal{U}\big([0,1]^d\big)\big) \\ &\leq \frac{k+1}{k}\, m^{1/d}\, d_{m,p}\big(\mathcal{U}\big([0,1]^d\big)\big), \end{aligned}$$

which yields for every integer $m \geq 1$

$$\limsup_{n\to\infty} n^{1/d}\, d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) \leq m^{1/d}\, d_{m,p}\big(\mathcal{U}\big([0,1]^d\big)\big),$$

since $\lim_{n\to\infty} k(n,m) = +\infty$. This finally implies

$$\lim_{n\to\infty} n^{1/d}\, d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) = \inf_{m\geq 0} m^{1/d}\, d_{m,p}\big(\mathcal{U}\big([0,1]^d\big)\big).$$

□

Proposition 11. *For every $p \geq 1$,*

$$Q^{dq}_{\|\cdot\|,p,d} = \lim_{n\to\infty} n^{1/d}\, d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) = \lim_{n\to\infty} n^{1/d}\, \bar{d}_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big)$$

Proof. Since we have $\bar{d}_{n,p}(X) \leq d_{n,p}(X)$ it remains to show

$$Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \leq \liminf_{n\to\infty} n^{1/d}\, \bar{d}_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big).$$

Let (Γ_n) be a sequence of optimal quantizers for $\bar{d}_{n,p}(\mathcal{U}\big([0,1]^d\big))$ and
For $0 < \varepsilon < 1/2$ let $C_\varepsilon = (1/2, \ldots, 1/2) + \frac{1-\varepsilon}{2}[-1,1]^d$ be the centered hypercube in $[0,1]^d$ with edge length $1-\varepsilon$ and midpoint $(1/2, \ldots, 1/2)$. Moreover let (Γ_n) be a sequence of quantizers such that

$$\bar{d}_p(\mathcal{U}\big([0,1]^d\big); \Gamma_n) \leq (1+\varepsilon)\bar{d}_{n,p}(\mathcal{U}\big([0,1]^d\big)).$$

Owing to Lemma 1, there is an integer $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n \geq n_\varepsilon, \quad C_\varepsilon \subset \mathrm{conv}(\Gamma_n).$$

We therefore get for any $n \geq n_\varepsilon$

$$(1+\varepsilon) d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) \geq d_p\big(\mathcal{U}\big([0,1]^d\big);\Gamma_n\big) \geq d_p\big(\mathcal{U}\big([0,1]^d\big)_{|\Gamma_n};\Gamma_n\big) \geq d_p\big(\mathcal{U}\big([0,1]^d\big)_{|C_\varepsilon};\Gamma_n\big).$$

Normalizing $\mathcal{U}\big([0,1]^d\big)_{|C_\varepsilon}$ into a probability distribution and applying Proposition 9, we get

$$(1+\varepsilon) d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big) \geq d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)_{|C_\varepsilon}\big) = (\lambda^d(C_\varepsilon))^{\frac{1}{p}} d_{n,p}\big(\mathcal{U}(C_\varepsilon)\big) = (1-\varepsilon)^{1+d/p} d_{n,p}\big(\mathcal{U}\big([0,1]^d\big)\big).$$

Hence, we obtain for all $0 < \varepsilon < 1/2$

$$\liminf_{n\to\infty} n^{1/d}\, d_{n,p}^p\big(\mathcal{U}\big([0,1]^d\big)\big) \geq \frac{(1-\varepsilon)^{1+d/p}}{1+\varepsilon}\, Q^{\mathrm{dq}}_{\|\cdot\|,p,d},$$

so that letting $\varepsilon \to 0$ yields the assertion. $\square$

Proposition 12. *Let* $\mathbf{P} = \sum_{i=1}^m s_i\, \mathcal{U}(C_i)$, $\sum_{i=1}^m s_i = 1$, $s_i > 0$, $i = 1,\ldots,m$, *where* $C_i = a_i + [0,l]^d$, $i = 1,\ldots,m$, *are pairwise disjoint hypercubes in* $\mathbb{R}^d$ *with common edge-length* l. *Set*

$$h := \frac{d\mathbf{P}}{d\lambda^d} = \sum_{i=1}^m s_i l^{-d} \mathbb{1}_{C_i}.$$

Then

(a) $\limsup_{n\to\infty} n^{1/d}\, d_{n,p}(\mathbf{P}) \leq Q^{dq}_{\|\cdot\|,p,d} \cdot \|h\|^{\frac{1}{p}}_{d/(d+p)}$.

(b) $\liminf_{n\to\infty} n^{1/d}\, \bar{d}_{n,p}(\mathbf{P}) \geq Q^{dq}_{\|\cdot\|,p,d} \cdot \|h\|^{\frac{1}{p}}_{d/(d+p)}$.

Proof. (a) For $n \in \mathbb{N}$, set

$$t_i = \frac{s_i^{d/(d+p)}}{\sum_{j=1}^m s_j^{d/(d+p)}} \qquad \text{and} \qquad n_i = \lfloor t_i n \rfloor,\ 1 \leq i \leq m.$$

Then, by Proposition 8 and Proposition 9, we get for every $n \geq \max_{1\leq i\leq m}(1/t_i)$

$$d_{n,p}^p(\mathbf{P}) \leq \sum_{i=1}^m s_i\, d_{n,p}^p(\mathcal{U}(C_i)) = l^p \sum_{i=1}^m s_i\, d_{n_i}^p(\mathcal{U}\big([0,1]^d\big)).$$

Proposition 10 then yields

$$n^{\frac{p}{d}}\, d_{n_i}^p(\mathcal{U}\big([0,1]^d\big)) = \left(\frac{n}{n_i}\right)^{\frac{p}{d}} n_i^{\frac{p}{d}}\, d_{n_i}^p(\mathcal{U}\big([0,1]^d\big)) \to t_i^{-\frac{p}{d}} Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \quad \text{ as } n \to \infty.$$

Noting that $\|h\|_{d/(d+p)} = l^p \left(\sum s_i^{d/(d+p)}\right)^{(d+p)/d}$, we get

$$\limsup_{n\to\infty} n^{\frac{p}{d}}\, d_{n,p}^p(\mathbf{P}) \leq Q^{\mathrm{dq}}_{\|\cdot\|,p,d} l^p \sum_{i=1}^m s_i\, t_i^{-\frac{p}{d}} \leq Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \cdot \|h\|_{d/(d+p)}.$$

(b) Let $\varepsilon \in (0, l/2)$ and denote by $C_{i,\varepsilon}$ the closed hypercube with the same center as C_i and edge-length $l - \varepsilon$. For $\alpha \in (0, \varepsilon/2)$, we denote by $\gamma_{\alpha_n,\varepsilon,i}$ the lattice with edge-length $\tilde{\alpha} = \frac{l}{\lceil l/\alpha \rceil}$ covering $C_i \setminus C_{i,\varepsilon}$.
We then get for every $i \in \{1, \dots, m\}$

$$|\gamma_{\alpha_n,\varepsilon,i}| = \Big(\frac{l}{\tilde{\alpha}} + 1\Big)^d - \Big(\frac{l}{\tilde{\alpha}} - 2\Big\lceil \frac{\varepsilon/2}{\tilde{\alpha}} \Big\rceil - 1\Big)^d$$

and define for every $\varepsilon \in (0, l/2), \alpha \in (0, \varepsilon/2)$

$$g_{l,\varepsilon}(\alpha) = \alpha^d |\gamma_{\alpha_n,\varepsilon,i}| = \Big(\alpha \frac{l}{\tilde{\alpha}} + \alpha\Big)^d - \Big(\alpha \frac{l}{\tilde{\alpha}} - 2\alpha \Big\lceil \frac{\varepsilon/2}{\tilde{\alpha}} \Big\rceil - \alpha\Big)^d.$$

Since $\frac{\alpha}{\tilde{\alpha}} \to 1$ and $2\alpha \Big\lceil \frac{\varepsilon/2}{\tilde{\alpha}} \Big\rceil \to \varepsilon$ as $\alpha \to 0$, we conclude

$$\forall \varepsilon \in (0, l/2), \quad \lim_{\alpha \to 0} g_{l,\varepsilon}(\alpha) = l^d - (l - \varepsilon)^d. \tag{7}$$

Let $\eta \in (0,1)$ and denote by Γ_n a sequence of n-quantizers such that $\bar{d}^p(\mathbf{P}; \Gamma_n) \leq (1+\eta) d_n^p(\mathbf{P})$. It follows from Proposition 7 that $\bar{d}^p(\mathbf{P}; \Gamma_n) \to 0$ for $n \to \infty$ so that Lemma 1 yields the existence of $n_\varepsilon \in \mathbb{N}$ such that for any $n \geq n_\varepsilon$

$$\bigcup_{1 \leq i \leq m} C_{i,\varepsilon} \subset \operatorname{conv}(\Gamma_n).$$

We then derive from Lemma 2

$$\begin{aligned}
\bar{d}^p(\mathcal{U}(C_i); \Gamma_n) &= l^{-d} \int_{C_i} \bar{F}^p(\xi; \Gamma_n)\, \lambda^d(d\xi) \\
&\geq l^{-d} \int_{C_{i,\varepsilon}} \bar{F}^p(\xi; \Gamma_n)\, \lambda^d(d\xi) = l^{-d} \int_{C_{i,\varepsilon}} F^p(\xi; \Gamma_n)\, \lambda^d(d\xi) \\
&\geq \frac{l^{-d}\,(l-\varepsilon)^d}{(1+\eta)^p}\, d^p\big(\mathcal{U}(C_{i,\varepsilon}); (\Gamma_n \cap \mathring{C}_i) \cup \gamma_{\alpha,\varepsilon,i}\big) - l^{-d}\,(l-\varepsilon)^d \Big(1 + \frac{1}{\eta}\Big)^p \cdot C_{\|\cdot\|} \cdot \alpha^p.
\end{aligned}$$

At this stage, we set for every $\rho > 0$

$$\alpha = \alpha_n = \Big(\frac{m}{\rho n}\Big)^{1/d} \tag{8}$$

and denote

$$n_i = |(\Gamma_n \cap \mathring{C}_i) \cup \gamma_{\alpha_n,\varepsilon,i}|.$$

Since $d_{n_i,p}(\mathcal{U}(C_{i,\varepsilon})) = (l - \varepsilon) d_{n_i,p}(\mathcal{U}([0,1]^d))$ owing to Proposition 9, we get

$$\begin{aligned}
n^{\frac{p}{d}} \bar{d}^p_{n,p}(\mathbf{P}) &\geq \frac{1}{1+\eta} \sum_{i=1}^m s_i\, n^{\frac{p}{d}}\, \bar{d}^p(\mathcal{U}(C_i); \Gamma_n) \\
&\geq \frac{l^{-d}\,(l-\varepsilon)^d}{(1+\eta)^{p+1}} \sum_{i=1}^m s_i\, n^{\frac{p}{d}}\, d^p\big(\mathcal{U}(C_{i,\varepsilon}); (\Gamma_n \cap \mathring{C}_i) \cup \gamma_{\alpha_n,\varepsilon,i}\big) \\
&\qquad - l^{-d}\,(l-\varepsilon)^d \frac{(1+\eta)^{p+1}}{\eta^p} \sum_{i=1}^m s_i \cdot C_{\|\cdot\|} \cdot \alpha^p \cdot n^{\frac{p}{d}} \\
&\geq \frac{l^{-d}\,(l-\varepsilon)^{d+p}}{(1+\eta)^{p+1}} \sum_{i=1}^m s_i\, n^{\frac{p}{d}}\, d^p_{n_i}\big(\mathcal{U}([0,1]^d)\big) - l^{-d}\,(l-\varepsilon)^d \frac{(1+\eta)^{p+1}}{\eta^p} \cdot C_{\|\cdot\|} \cdot \Big(\frac{m}{\rho}\Big)^{\frac{p}{d}}.
\end{aligned} \tag{9}$$

Since

$$\frac{n_i}{n} \le \frac{|\Gamma_n \cap \mathring{C}_i|}{n} + \frac{g_{l,\varepsilon}(\alpha_n)}{n\alpha^d} = \frac{|\Gamma_n \cap \mathring{C}_i|}{n} + \frac{\rho}{m} g_{l,\varepsilon}(\alpha_n),$$

we conclude from (7) and (8)

$$\limsup_{n\to\infty} \sum_{i=1}^m \frac{n_i}{n} \le 1 + \rho\big(l^d - (l-\varepsilon)^d\big).$$

We may choose a subsequence (still denoted by (n)), such that

$$n^{1/d}\, \bar{d}_{n,p}(\mathbf{P}) \to \liminf_{n\to\infty} n^{1/d}\, d_{n,p}(\mathbf{P}) \qquad \text{and} \qquad \frac{n_i}{n} \to v_i \in [0, 1 + \rho(l^d - (l-\varepsilon)^d)].$$

As a matter of fact, it holds $v_i > 0$, $1 \le i \le m$: otherwise Proposition 10 would yield

$$\begin{aligned} n^{\frac{p}{d}}\, \bar{d}^p_{n,p}(\mathbf{P}) \ge & \frac{l^{-d}\,(l-\varepsilon)^{d+p}}{(1+\eta)^{p+1}} \sum_{i=1}^m s_i \left(\frac{n_i}{n}\right)^{-\frac{p}{d}} n_i^{\frac{p}{d}}\, d^p_{n_i}\big(\mathcal{U}\big([0,1]^d\big)\big) \\ & - l^{-d}\,(l-\varepsilon)^d \frac{(1+\eta)^{p+1}}{\eta^p} \cdot C_{\|\cdot\|} \cdot \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \\ & \to +\infty \end{aligned}$$

which contradicts (a). Consequently, we may normalize the v_i's by setting

$$\widetilde{v}_i = \frac{v_i}{1 + \rho(l^d - (l-\varepsilon)^d)}$$

so that $\sum_{i=1}^m \widetilde{v}_i \le 1$. We conclude from Proposition 10

$$\begin{aligned} \liminf_{n\to\infty} \sum_{i=1}^m s_i\, n^{\frac{p}{d}}\, d^p_{n_i}\big(\mathcal{U}\big([0,1]^d\big)\big) &= \sum_{i=1}^m s_i \lim_{n\to\infty} \left(\frac{n\big(1+\rho(l^d-(l-\varepsilon)^d)\big)}{n_i}\right)^{\frac{p}{d}} n_i^{\frac{p}{d}}\, d^p_{n_i}\big(\mathcal{U}\big([0,1]^d\big)\big) \\ &= Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \sum_{i=1}^m s_i\, \widetilde{v}_i^{\,-\frac{p}{d}} \\ &\ge Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \inf_{\sum_i y_i \le 1, y_i \ge 0} \sum_{i=1}^m s_i y_i^{-\frac{p}{d}} \\ &= Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \left(\sum_{i=1}^m s_i^{d/(d+p)}\right)^{(d+p)/d}. \end{aligned}$$

Hence, we derive from (9)

$$\begin{aligned} \liminf_{n\to\infty} n^{\frac{p}{d}}\, \bar{d}^p_{n,p}(\mathbf{P}) \ge & \frac{l^{-d}\,(l-\varepsilon)^{d+p}}{(1+\eta)^{p+1}\big(1+\rho(l^d-(l-\varepsilon)^d)\big)^{\frac{p}{d}}}\, Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \left(\sum_{i=1}^m s_i^{d/(d+p)}\right)^{(d+p)/d} \\ & - l^{-d}\,(l-\varepsilon)^d \frac{(1+\eta)^{p+1}}{\eta^p} \cdot C_{\|\cdot\|} \cdot \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \end{aligned}$$

so that sending $\varepsilon \to 0$ implies

$$\begin{aligned} \liminf_{n\to\infty} n^{\frac{p}{d}}\, \bar{d}^p_{n,p}(\mathbf{P}) &\ge \frac{l^p}{(1+\eta)^{p+1}}\, Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \left(\sum_{i=1}^m s_i^{d/(d+p)}\right)^{(d+p)/d} - \frac{(1+\eta)^{p+1}}{\eta^p} C_{\|\cdot\|} \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \\ &= \frac{1}{(1+\eta)^{p+1}}\, Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \cdot \|h\|_{d/(d+p)} - \frac{(1+\eta)^{p+1}}{\eta^p} C_{\|\cdot\|} \left(\frac{m}{\rho}\right)^{\frac{p}{d}} \end{aligned}$$

and, finally, letting successively ρ go to ∞ and η go to 0 yields the assertion. □

Proposition 13. *Assume that* $\mathbf{P}$ *is absolutely continuous w.r.t.* λ^d *with compact support. Then*

(a) $$\limsup_{n\to\infty} n^{\frac{p}{d}}\, d_{n,p}(\mathbf{P}) \le Q^{dq}_{\|\cdot\|,p,d} \cdot \|h\|^{\frac{1}{p}}_{d/(d+p)}.$$

(b) $$\liminf_{n\to\infty} n^{\frac{p}{d}}\, \bar{d}_{n,p}(\mathbf{P}) \ge Q^{dq}_{\|\cdot\|,p,d} \cdot \|h\|^{\frac{1}{p}}_{d/(d+p)}.$$

Proof. Let $C = [-l/2, l/2]^d$ be a closed hyper hypercube centered at the origin, parallel to the coordinate axis with edge-length l, such that $\mathrm{supp}(\mathbf{P}) \subset C$. For $k \in \mathbb{N}$ consider the tessellation of C into k^d closed hypercubes with common edge-length l/k. To be precise, for every $\underline{i} = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, we set

$$C_{\underline{i}} = \prod_{r=1}^{d} \Big[-\frac{l}{2} + \frac{i_r l}{k}, -\frac{l}{2} + \frac{(i_r+1)l}{k} \Big].$$

Set $h = \frac{d\mathbf{P}}{d\lambda^d}$ and

$$\mathbf{P}_k = \sum_{\substack{\underline{i}\in\mathbb{Z}^d \\ 0\le i_r<k}} \mathbf{P}(C_{\underline{i}})\,\mathcal{U}(C_{\underline{i}}), \qquad h_k = \frac{d\mathbf{P}_k}{d\lambda^d} = \sum_{\substack{\underline{i}\in\mathbb{Z}^d \\ 0\le i_r<k}} \frac{\mathbf{P}(C_{\underline{i}})}{\lambda^d(C_{\underline{i}})} \mathbb{1}_{C_{\underline{i}}}. \tag{10}$$

By differentiation of measures we obtain $h_k \to h$, λ^d-*a.s.* as $k \to \infty$. Which in turn implies, owing to Scheffé's Lemma,

$$\lim_{k\to\infty} \|h_k - h\|_1 = 0$$

and

$$\lim_{k\to\infty} \|h_k\|_{d/(d+p)} = \|h\|_{d/(d+p)}$$

s ince $\|h_k - h\|_{d/(d+p)} \le \Big(\lambda^d(C)\Big)^{\frac{p}{d}} \|h_k - h\|_1$ by Jensen's Inequality applied to the probability measure $\frac{\lambda_{d\,|C}}{\lambda_d(C)}$. Moreover, by Proposition 12 we have

$$\lim_{n\to\infty} n^{1/d}\, d_{n,p}(\mathbf{P}_k) = Q^{\mathrm{dq}}_{\|\cdot\|,p,d}\, \|h_k\|^{\frac{1}{p}}_{d/(d+p)}. \tag{11}$$

Likewise, we define an inner approximation of $\mathbf{P}$: Denote by

$$C^k = \bigcup_{\underbrace{C_{\underline{i}} \subset \mathring{\mathrm{supp}}(\mathbf{P})}} C_{\underline{i}}$$

the union of the hypercubes $C_{\underline{i}}$ lying in the interior of $\mathrm{supp}(\mathbf{P})$. Setting

$$\mathring{\mathbf{P}}_k = \sum_{\underbrace{C_{\underline{i}} \subset \mathring{\mathrm{supp}}(\mathbf{P})}} \mathbf{P}(C_{\underline{i}})\,\mathcal{U}(C_{\underline{i}}),$$

$$\mathring{h}_k = \frac{d\mathring{\mathbf{P}}_k}{d\lambda^d} = h_k \mathbb{1}_{C^k},$$

we have as above that

$$\mathring{h}_k \to h, \quad \lambda^d\text{-a.s.} \quad \text{for } k \to \infty.$$

Consequently

$$\lim_{k\to\infty} \|\mathring{h}_k - h\|_1 = 0 \quad \text{and} \quad \lim_{k\to\infty} \|\mathring{h}_k\|_{d/(d+p)} = \|h\|_{d/(d+p)}.$$

We get likewise by Proposition 12 that, for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{1/d}\, d_{n,p}(\mathring{\mathbf{P}}_k) = Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \cdot \|\mathring{h}_k\|^{\frac{1}{p}}_{d/(d+p)}. \tag{12}$$

(a) Let $0 < \varepsilon < 1$ and $n \geq 2^d/\varepsilon$. If we divide each edge of the hypercube C into

$$m = \left\lfloor (\varepsilon n)^{1/d} \right\rfloor - 1$$

intervals of equal length l/m, the interval endpoints define $m+1$ grid points on each edge. Denoting by $\Gamma_1 = \Gamma_1(\varepsilon, n)$ the product quantizer made up by this procedure, we clearly have

$$|\Gamma_1| = (m+1)^d = \left\lfloor (\varepsilon n)^{1/d} \right\rfloor^d =: n_1.$$

For this product quantizer it follows from Propositions 5 and 4 that, for all $\xi \in C$,

$$\begin{aligned} F^p(\xi;\Gamma_1) &\leq C_{\|\cdot\|} \sum_{i=1}^{d} \left(\frac{l}{2m}\right)^p \\ &\leq C_{\|\cdot\|,p,d}\, \frac{l^p}{(\varepsilon n)^{\frac{p}{d}}}. \end{aligned}$$

For $n_2 = \lfloor (1-\varepsilon)n \rfloor$ let Γ_2 be a n_2-quantizer such that $d^p(\mathbf{P}_k;\Gamma_2) \leq (1+\varepsilon) d^p_{n_2}(\mathbf{P}_k)$. We clearly have $|\Gamma_1 \cup \Gamma_2| \leq n$ and

$$\begin{aligned} n^{\frac{p}{d}} \left| \int F^p(\xi;\Gamma_1\cup\Gamma_2) d\mathbf{P}_k(\xi) - \int F^p(\xi;\Gamma_1\cup\Gamma_2) d\mathbf{P}(\xi) \right| & \\ \leq n^{\frac{p}{d}} \int F^p(\xi;\Gamma_1\cup\Gamma_2) |h_k(\xi) - h(\xi)| d\lambda^d \xi & \\ \leq C_{\|\cdot\|,p,d}\, \frac{l^p}{\varepsilon^{\frac{p}{d}}} \|h_k - h\|_1 & \\ = c_{1,\varepsilon} \|h_k - h\|_1 & \end{aligned}$$

for $k \in \mathbb{N}$ and $n \geq \max\left\{\frac{2^d}{\epsilon}, \frac{1}{1-\varepsilon}\right\}$. This implies

$$\begin{aligned} n^{\frac{p}{d}} d^p_{n,p}(\mathbf{P}) &\leq n^{\frac{p}{d}} \int F^p(\xi;\Gamma_1\cup\Gamma_2) d\mathbf{P}(\xi) \\ &\leq n^{\frac{p}{d}} \int F^p(\xi;\Gamma_1\cup\Gamma_2) d\mathbf{P}_k(\xi) + c_1 \|h_k - h\|_1 \\ &\leq n^{\frac{p}{d}} \int F^p(\xi;\Gamma_2) d\mathbf{P}_k(\xi) + c_1 \|h_k - h\|_1 \\ &\leq (1+\varepsilon)\, n^{\frac{p}{d}} d^p_{n_2}(\mathbf{P}_k) + c_{1,\varepsilon} \|h_k - h\|_1, \end{aligned}$$

so that we conclude from (11)

$$\limsup_{n\to\infty} n^{\frac{p}{d}} d^p_{n,p}(\mathbf{P}) \leq \frac{1+\varepsilon}{(1-\varepsilon)^{\frac{p}{d}}} (Q^{\mathrm{dq}}_{\|\cdot\|,p,d})^p \|h_k\|_{d/(d+p)} + c_{1,\varepsilon} \|h_k - h\|_1.$$

Letting first k go to infinity and then letting ε go to zero yields

$$\limsup_{n\to\infty} n^{1/d} d_{n,p}(\mathbf{P}) \leq Q^{\mathrm{dq}}_{\|\cdot\|,p,d} \|h_k\|^{\frac{1}{p}}_{d/(d+p)}.$$

(b) Assume that Γ_3 is an n_2-quantizer such that $\bar{d}^p(\mathbf{P};\Gamma_3) \leq (1+\varepsilon)\,\bar{d}^p_{n_2}(\mathbf{P})$. Again it holds $|\Gamma_1 \cup \Gamma_3| \leq n$ and we derive as above

$$n^{\frac{p}{d}}\left|\int F^p(\xi;\Gamma_1\cup\Gamma_3)d\mathring{\mathbf{P}}_k(\xi) - \int F^p(\xi;\Gamma_1\cup\Gamma_3)d\mathbf{P}(\xi)\right| \leq c_{2,\varepsilon}\|\mathring{h}_k - h\|_1. \tag{13}$$

Moreover, Lemma 1 yields for every $k\in\mathbb{N}$ the existence of $n_{k,\varepsilon}\in\mathbb{N}$ such that, for all $n \geq n_{k,\varepsilon}$,

$$\begin{aligned}(1+\varepsilon)\,\bar{d}^p_{n_2,p}(\mathbf{P}) \geq \bar{d}^p_p(\mathbf{P};\Gamma_3) &\geq \int_{\mathrm{conv}(\Gamma_3)} F^p(\xi;\Gamma_3)d\mathbf{P}(\xi)\\ &\geq \int_{C^k} F^p(\xi;\Gamma_3)d\mathbf{P}(\xi) \geq \int_{C^k} F^p(\xi;\Gamma_1\cup\Gamma_3)d\mathbf{P}(\xi).\end{aligned}$$

Thus, we derive from (13) that, for every $n \geq \max\left(n_{k,\varepsilon}, \frac{2^d}{\varepsilon}, \frac{1}{1-\varepsilon}\right)$,

$$\begin{aligned}(1+\varepsilon)\,n^{\frac{p}{d}}\,\bar{d}^p_{n_2,p}(\mathbf{P}) &\geq n^{\frac{p}{d}}\int_{C^k} F^p(\xi;\Gamma_1\cup\Gamma_3)d\mathbf{P}(\xi)\\ &\geq n^{\frac{p}{d}}\int_{C^k} F^p(\xi;\Gamma_1\cup\Gamma_3)d\mathring{\mathbf{P}}_k(\xi) - c_{2,\varepsilon}\|\mathring{h}_k - h\|_1\\ &\geq n^{\frac{p}{d}}\,d^p_{n,p}(\mathring{\mathbf{P}}_k) - c_{2,\varepsilon}\|\mathring{h}_k - h\|_1,\end{aligned}$$

which yields, once combined with (12),

$$\frac{1+\varepsilon}{(1-\varepsilon)^{\frac{p}{d}}}\liminf_{n\to\infty} n_2^{\frac{p}{d}}\,\bar{d}^p_{n_2,p}(\mathbf{P}) \geq Q^{\mathrm{dq}}_{\|\cdot\|,p,d}\|\mathring{h}_k\|_{d/(d+p)} - c_{2,\varepsilon}\|\mathring{h}_k - h\|_1.$$

Letting first k go to ∞ and then letting ε go to 0, we get

$$\liminf_{n\to\infty} n^{\frac{1}{d}}\,\bar{d}_{n,p}(\mathbf{P}) \geq Q^{\mathrm{dq}}_{\|\cdot\|,p,d}\|h\|^{\frac{1}{p}}_{d/(d+p)}.$$

□

Proposition 14 (Singular distribution)**.** *Assume that* $\mathbf{P}$ *is singular with respect to* λ^d *and has compact support. Then*

$$\limsup_{n\to\infty} n^{\frac{p}{d}}\,\bar{d}_{n,p}(\mathbf{P}) = 0.$$

Proof. We follow the lines of Step 4 in Graf and Luschgy's proof of Zador's Theorem (see [4]). Let A be a Borel set such that $\mathbf{P}(A) = 1$ and $\lambda^d(A) = 0$. Let $\varepsilon > 0$; by the outside regularity of λ^d there exists an open set $O = O(\varepsilon)$ set such that $\lambda^d(O) \leq \varepsilon$ (and $\mathbf{P}(O) = 1$). Let C be an open hypercube with edges of length ℓ, parallel to the coordinate axis containing the closure of A.
Let $C_k = \prod_{i=1}^d [c_{k,i}, c_{k,i}+\ell_i)$, $k\in\mathbb{N}$ be a countable partition of A consisting of nonempty half-open hypercubes, still with edges parallel to the coordinate axis (see, *e.g.* Lemma 1.4.2 in [3]).
Let $m = m(\varepsilon)\in\mathbb{N}$ such that $\displaystyle\sum_{k\geq m+1}\mathbf{P}(C_k) \leq \varepsilon^{\frac{p}{d}}\ell^{-p}$.
Let $n\in\mathbb{N}$, $n \geq 2^{d+1}$ and let $n_1,\ldots,n_d \geq 2$ be integers such that the product $n_1^d+\cdots+n_m^d \leq n/2$. One designs a grid Γ as follows.
For every $k\in\{1,\ldots,m\}$, we consider the lattice of C_k of size n_i^d defined by

$$\prod_{i=1}^d\left\{c_{k,i} + \frac{r_i}{n_k-1}\ell_i,\ r_i = 0,\ldots,n_k-1,\ i = 1,\ldots,d\right\}.$$

Then one defines likewise the lattice of C of size $n_{m+1}^d \le n/2$

$$\prod_{i=1}^{d} \Big\{ c_{k,i} + \frac{r_i}{n_{m+1}-1}\ell_i,\ r_i = 0,\dots,n_{m+1}-1,\ i = 1,\dots,d \Big\}.$$

The grid Γ is made up with all the points of the $m+1$ above finite lattices.
Now let $\xi \in A$. It is clear from the definition of the function F_p that

$$F_p(\xi;\Gamma) \le \begin{cases} C_{\|.\|}\big(\ell_k/n_k\big)^p & \text{if } \xi \in \bigcup_{k=1}^m C_k \\ C_{\|.\|}\big(\ell/n_{m+1}\big)^p & \text{if } \xi \in C \setminus \bigcup_{k=1}^m C_k \end{cases}$$

where $C_{\|.\|} > 0$ is a real constant only depending on the norm.
As a consequence

$$\begin{aligned} d_{n,p}^p(\mathbf{P}) &= \sum_{k=1}^m \int_{C_k} F^p(\xi;\Gamma)dP(\xi) + \int_{C\setminus\bigcup_{k=1}^m C_k} F^p(\xi;\Gamma)dP(\xi) \\ &\le C_{\|.\|}\Big(\sum_{k=1}^m (\ell_k/n_k)^p P(C_k) + (\ell/n_{m+1})^p P(C\setminus \bigcup_{k=1}^m C_k)\Big). \end{aligned}$$

Set for every $k \in \{1,\dots,m\}$, $n_k = \left\lfloor \dfrac{\ell_k (n/2)^{\frac1d}}{(\sum_{k'=1}^d \ell_{k'}^d)^{\frac1d}} \right\rfloor$ and $n_{m+1} = \lfloor (n/2)^{\frac1d} \rfloor$. Note that

$$\sum_{k'=1}^d \ell_{k'}^d = \sum_{k=1}^m \lambda^d(C_k) \le \lambda^d(A) \le \varepsilon.$$

Elementary computations show that for large enough n, all the integers n_k are greater than 1 and that

$$\sum_{k=1}^m (\ell_k/n_k)^p P(C_k) + (\ell/n_{m+1})^p P(C\setminus\bigcup_{k=1}^m C_k) \le (\sum_{k'=1}^d \ell_{k'}^d)^{\frac{p}{d}} (n/2)^{-\frac{p}{d}} \mathbf{P}\big(\cup_{1\le k\le m} C_k\big) + (n/2)^{-\frac{p}{d}} \ell^p \mathbf{P}\big(C\setminus\bigcup_{k=1}^m C_k\big)$$

so that

$$\limsup_n n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) \le C_{\|.\|}(\varepsilon/2)^{\frac{p}{d}}$$

which in turn implies by letting ε go to 0 that

$$\limsup_n n^{\frac{p}{d}} d_{n,p}^p(\mathbf{P}) = 0.$$

□

Proof of Theorem 2: The assertion (a) follows directly from Propositions 13 and 14 and the fact that it holds $\bar{d}_{n,p}(X) \le d_{n,p}(X)$ for every $n \in \mathbb{N}$. Furthermore, part (c) was derived in [9], Section 5.1. Hence, it remains to prove (b).

Proof. Step 1. (Lower bound) If X is compactly supported, the assertion follows from Proposition 13. Otherwise, set for every $R \in (0,\infty)$,

$$C_R = [-R,R]^d$$

and for $k \in \mathbb{N}$

$$\mathbf{P}(\cdot|C_k) = \frac{h\mathbb{1}_{C_k}}{\mathbf{P}(C_k)}\lambda^d.$$

Proposition 13 yields again

$$\lim_{n\to\infty} n^{\frac{1}{d}}\,\bar d_{n,p}(\mathbf{P}(\cdot|C_k)) = Q^{\mathrm{dq}}_{\|\cdot\|,p,d}\cdot \|h\mathbb{1}_{C_k}/\mathbf{P}(C_k)\|^{\frac{1}{p}}_{d/(d+p)}, \tag{14}$$

so that $\bar d^p_{n,p}(\mathbf{P}) \geq \mathbf{P}(\cdot|C_k)\bar d^p_{n,p}(\mathbf{P}(\cdot|C_k))$ implies for all $k\in\mathbb{N}$

$$\liminf_{n\to\infty} n^{\frac{1}{d}}\,\bar d_{n,p}(\mathbf{P}) \geq Q^{\mathrm{dq}}_{\|\cdot\|,p,d}\cdot \|h\mathbb{1}_{C_k}\|^{\frac{1}{p}}_{d/(d+p)}.$$

Sending k to infinity, we arrive at

$$\liminf_{n\to\infty} n^{\frac{1}{d}}\,\bar d_{n,p}(\mathbf{P}) \geq Q^{\mathrm{dq}}_{\|\cdot\|,p,d}\cdot \|h\|^{\frac{1}{p}}_{d/(d+p)}.$$

Step 2 (*Upper bound*, $\mathrm{supp}(\mathbf{P})=\mathbb{R}^d$). Let $\rho\in(0,1)$. Set $K=C_{k+\rho}$ and $K_\rho=C_k$. Let $\Gamma_{k,\alpha,\rho}$ be the lattice grid associated to $K\setminus K_\rho$ with edge $\alpha>0$ as defined in the remark that follows the "firewall" Lemma 2. It is straightforward that there exists a real constant $C>0$ such that

$$\forall\, k\in\mathbb{N}, \forall\,\rho\in(0,1), \forall\,\alpha\in(0,\rho):\quad |\Gamma_{\alpha,\rho}|\leq Cd\rho k^{d-1}\alpha^{-d}.$$

Let $\varepsilon\in(0,1)$. For every $n\geq 1$, set $\alpha_n=\tilde\alpha_0 n^{-\frac{1}{d}}$ where $\tilde\alpha_0\in(0,1)$ is a real constant and

$$n_0=|\Gamma_{k,\alpha_n,\rho}|,\quad n_1=\lfloor(1-\varepsilon)(n-n_0)\rfloor,\quad n_2=\lfloor\varepsilon(n-n_0)\rfloor,$$

so that $\alpha_n\in(0.\rho)$, $n_0+n_1+n_2\leq n$ and $n_i\geq 1$ for large enough n. For every $\xi\in K_\rho=C_k$, for every grid $\Gamma\subset\mathbb{R}^d$, we know by the "firewall" Lemma lem:firewall that

$$F^p(\xi;(\Gamma\cap\mathring{K})\cup\Gamma_{\alpha,\rho})\leq(1+\eta)^pF^p(\xi;\Gamma)+(1+\eta)^p(1+1/\eta)^pC_{\|.\|}\alpha^p.$$

Let $\Gamma_1=\Gamma_1(n_1,k)$ be a n_1 quantizer such that $d^p_{n_1}(\mathbf{P}(.|C_k);\Gamma_1)\leq(1+\eta)d^p_{n_1}(\mathbf{P}(.|C_k))$. Set $\Gamma_1'=((\Gamma_1\cap\mathring{C}_{k+\rho})\cup\Gamma_{k,\alpha_n,\rho})$. One has $\Gamma_1'\subset C_{k+2\rho}$ for large enough n (so that $\alpha_n<\rho$).
Let moreover $\Gamma_2=\Gamma_2(n_2,k)$ be a n_2 quantizer such that $\bar d^p_{n_2}(\mathbf{P}(.|C_k^c);\Gamma_2)\leq(1+\eta)\bar d^p_{n_2}(\mathbf{P}(.|C_k^c))$. For $n\geq n_\rho$, we may assume that $C_{k+2\rho}\subset\mathrm{conv}\,\Gamma_2$ owing to Lemma 1 since $C_{k+2\rho}=\mathrm{conv}(C_{k+2\rho}\setminus C_{k+\frac{3}{2}\rho})$ and $C_{k+2\rho}\setminus C_{k+\frac{3}{2}\rho}\subset\overbrace{\mathrm{supp}\mathbf{P}(.|C_k^c)}^{\circ}$. As a consequence $\Gamma_1'\subset\mathrm{conv}(\Gamma_2)$ so that $\mathrm{conv}(\Gamma_1')\subset\mathrm{conv}(\Gamma_2)=\mathrm{conv}(\Gamma)$ where $\Gamma=\Gamma_1'\cup\Gamma_2$and

$$C_{k+\rho}\subset\mathrm{conv}(\Gamma)=\mathrm{conv}(\Gamma_2).$$

Now

$$\begin{aligned}\bar d^p_{n,p}(\mathbf{P}) &\leq \int_{C_k}\left(F^p(\xi;\Gamma)\mathbf{1}_{\{\xi\in\mathrm{conv}(\Gamma_2)\}}+\underbrace{d(\xi,\Gamma)^p\mathbf{1}_{\{\xi\notin\mathrm{conv}(\Gamma_2)\}}}_{=0}\right)d\mathbf{P}(\xi)\\ &\quad+\int_{C_k^c}\left(F^p(\xi;\Gamma)\mathbf{1}_{\{\xi\in\mathrm{conv}(\Gamma_2)\}}+d(\xi,\Gamma)^p\mathbf{1}_{\{\xi\notin\mathrm{conv}(\Gamma_2)\}}\right)d\mathbf{P}(\xi).\end{aligned}$$

Using that, for every $\xi\in C_k$,

$$\begin{aligned}F^p(\xi;\Gamma) &\leq F^p(\xi;\Gamma_1')\\ &\leq (1+\eta)^p\Big(F^p(\xi;\Gamma_1)+(1+1/\eta)^p\,C_{\|.\|}\,\alpha_n^p\Big)\end{aligned}$$

implies

$$\begin{aligned}\bar d^p_{n,p}(\mathbf{P}) &\leq \mathbf{P}(C_k)(1+\eta)^p\Big((1+\eta)\,d^p_{n_1,p}(\mathbf{P}(.|C_k))+(1+1/\eta)^p\,C_{\|.\|}\,\tilde\alpha_0\,n^{-\frac{1}{d}}\Big)\\ &\quad+\mathbf{P}(C_k^c)\,(1+\eta)\,\bar d_{n_2,p}(\mathbf{P}(.|C_k^c)).\end{aligned}$$

Consequently

$$\begin{aligned} n^{\frac{p}{d}}\bar{d}^p_{n,p}(\mathbf{P}) &\leq \mathbf{P}(C_k)(1+\eta)^p\left[(1+\eta)\left(\frac{n}{n_1}\right)^{\frac{p}{d}} n_1^{\frac{p}{d}}\, d^p_{n_1,p}(\mathbf{P}(.|C_k)) + (1+1/\eta)^p C_{\|.\|}\tilde{\alpha}_0\right] \\ &\quad +(1+\eta)\left(\frac{n}{n_2}\right)^{\frac{p}{d}} \mathbf{P}(C_k^c)\, n_2^{\frac{p}{d}}\, \bar{d}_{n_2,p}(\mathbf{P}(.|C_k^c)) \end{aligned}$$

which in turn implies, using the d-dimensional version of the extended Pierce Lemma (Proposition 7),

$$\begin{aligned} \limsup_n n^{\frac{p}{d}}\bar{d}^p_{n,p}(\mathbf{P}) &\leq \mathbf{P}(C_k)(1+\eta)^p\left(\left(\frac{(1+\eta)^{-p/d}}{(1-\varepsilon)(1-Cd\rho k^{d-1}\tilde{\alpha}_0^{-d})}\right)^{\frac{p}{d}} Q^{dq}_{\|.\|}\|h\mathbf{1}_{C_k}\|_{L^{\frac{d}{d+p}}}\right. \\ &\quad \left. +(1+1/\eta)^p C_{\|.\|}\tilde{\alpha}_0\right) \\ &\quad +\mathbf{P}(C_k^c)\,(1+\eta)\; C_{p,d}\,\|X\mathbf{1}_{\{X\in C_k^c\}}\|^p_{L^{p+\delta}}\left(\frac{1}{\varepsilon(1-Cd\rho k^{d-1}\tilde{\alpha}_0^{-d})}\right)^{\frac{p}{d}}. \end{aligned}$$

One concludes by letting successively ρ, $\tilde{\alpha}_0$ and η go to 0, sending $k\to\infty$ and finally ε to 0.

Step 3. (Upper bound: general case). Let $\rho\in(0,1)$. Set $\mathbf{P}_\rho=\rho\mathbf{P}+(1-\rho)\mathbf{P}_0$ where $\mathbf{P}_0=\mathcal{N}(0;I_d)$ (d-dimensional normal distribution). It is clear from the very definition of $\bar{d}_{n,p}$ that $\bar{d}_{n,p}(\mathbf{P})\leq\frac{1}{\rho}\bar{d}_{n,p}(\mathbf{P}_\rho)$ since $\mathbf{P}\leq\frac{1}{\rho}\leq\mathbf{P}_\rho$. The distribution $\mathbf{P}_\rho$ has $h_\rho=\rho h+(1-\rho)h_0$ (with obvious notations) and one concludes by noting that

$$\lim_{\rho\to 0}\|h_\rho\|_{d/(d+p)}=\|h\|_{d/(d+p)}$$

owing to the Lebesgue dominated convergence Theorem. □

Proof of Proposition 1:

Proof. Using Hoelder's inequality one easily checks that for $0\leq r\leq p$ and $x\in\mathbb{R}^d$ it holds

$$|x|_{\ell^r}\leq d^{\frac{1}{r}-\frac{1}{p}}\,|x|_{\ell^p}.$$

Moreover, for $m\in\mathbb{N}$ set $n=m^d$ and let Γ' be an optimal quantizer for $d_{m,p}(\mathcal{U}([0,1]))$ (or at least $(1+\varepsilon)$-optimal for $\varepsilon>0$). Denoting $\Gamma=\prod_{i=1}^d\Gamma'$, it then follows from Proposition 5 that

$$n^{\frac{p}{d}}\,d^p_n(\mathcal{U}([0,1]^d))\leq n^{\frac{p}{d}}\,d^p(\mathcal{U}([0,1]^d);\Gamma)=m^p\sum_{i=1}^d d^p(\mathcal{U}([0,1]);\Gamma')=d\,m^p\,d^p_m(\mathcal{U}([0,1])).$$

Combining both results and reminding that $Q^{\mathrm{dq}}_{\|\cdot\|,p,d}$ holds as an infimum, we obtain for $r\in[0,p]$,

$$\left(Q^{\mathrm{dq}}_{|\cdot|_{\ell^r},p,d}\right)^p\leq d^{\frac{p}{r}-1}\,n^{\frac{p}{d}}\,d^p_{n,|\cdot|_{\ell^p}}(\mathcal{U}([0,1]^d))\leq d^{\frac{p}{r}}\,m^p\,d^p_m(\mathcal{U}([0,1])),$$

which finally proves the assertion by sending $m\to\infty$. □

6 Further remarks prospects

This result does not complete the theoretical investigations about dual quantization (beyond the existence of optimal dual quantizers in the case $p = 1$ left open in [9]): the first one is to elucidate the behaviour of the constant $Q^{dq}_{\|.\|,p,d}$ coming out in Theorem 2 as d goes to infinity, or at least that of the ratio $\frac{Q^{dq}_{\|.\|,p,d}}{Q^{vq}_{\|.\|,p,d}}$. From a practical point of view, is it possible to evaluate the mean dual quantization error induces by an optimal Voronoi quantization grid? An answer to that question would be very valuable for applications since many optimal quantization grids have been computed for various distributions (see *e.g.* [7] for Gaussian distributions). Can we preserve the above results (as well as existence of dually optimal grids) for unbounded r.v. when switching to another extension of the random splitting operator *outside* the convex hull of the grid?

Many natural questions solved in the optimal Voronoi quantization theory remain open. Among others "Is there a counterpart to the empirical measure theorem for (asymptotically) optimal quantizers?" (see Theorem in [4])? "How does dual quantization behave with respect to empirical distribution of i.i.d. n-samples of a given distribution?". Is it possible to develop an infinite dimensional "functional" dual quantization ?

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